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On Weight Functions and Norms of Some Singular Integral Operators

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Dedicated to Professor Takahiko Nakazi on the occasion of his 60th birthday

ABSTRACT

Let m be a normalized Lebesgue measure on the unit circle $\mathbf{T}$. Let a and b be bounded m-measurable functions on $\mathbf{T}$, and let w be a positive and integrable function on $\mathbf{T}$. It is well known that the Riesz projection P_+ is bounded on the weighted space $L^2(wdm)$ if and only if w has the Helson-Szegő representation. Let $P_- = I - P_+$, where I denotes the identity operator. In this paper, if the singular integral operator $aP_+ + bP_-$ is bounded and invertible on the weighted space $L^2(wdm)$, then we establish the Helson-Szegő type representation of the weight function w using the operator norms of $aP_+ + bP_-$ and $(aP_+ + bP_-)^{-1}$.

KEYWORDS: Singular integral operator, Riesz projection, Norm, Hardy space, Weight function, Helson-Szegő weight.

MSC (2000): Primary 45E10, 47B35; Secondary 46J15.

1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $\mathbf{T} = \{z\,;\,|z|=1\}$. That is, $dm(\zeta) = \dfrac{d\theta}{2\pi}$ for $\zeta = e^{i\theta}$. Throughout this paper, we always assume that the weight function w satisfies $w > 0$ a.e. on $\mathbf{T}$, and $w \in L^1(\mathbf{T}) = L^1(\mathbf{T}, dm)$. Let $\mathcal{P}$ denote the set of all trigonometric polynomials. Define the Riesz projection P_+ by

$$(P_+ f)(e^{it}) = \sum_{k \geq 0} \hat{f}(k) e^{ikt}, \quad f \in \mathcal{P},$$

where $\hat{f}(k)$ denotes the k-th Fourier coefficient of f. Let S be the singular integral operator (SIO) defined by

$$(Sf)(\zeta) = \frac{1}{\pi i} \int_{\mathbf{T}} \frac{f(z)}{z - \zeta} dz,$$

where the integral is understood in the sense of Cauchy's principal value (cf. [13], p.11). If f is in L^1, then $(Sf)(\zeta)$ exists for almost everywhere ζ on $\mathbf{T}$, and Sf becomes a measurable function on $\mathbf{T}$. Let $P_-=I-P_+$, where I denotes the identity operator. Then $P_+^2=P_+$, $P_-^2=P_-$, $P_+=\dfrac{I+S}{2}$, and $P_-=\dfrac{I-S}{2}$. Since $S=P_+-P_-$, $S^2=I$. The harmonic conjugate function $\tilde{f}$ of f is defined by

$$\tilde{f}(e^{i\theta})=\int_{-\pi}^{\pi}\cot\left(\frac{\theta-t}{2}\right)f(e^{it})\,\frac{dt}{2\pi},$$

where the integral is understood in the sense of Cauchy's principal value. The Hilbert transform H is defined by $H=-i(P_+-P_-)$. Hence $H=-iS$. Then

$$\begin{aligned}\tilde{f}&=-i(P_+f-P_-f)+i\hat{f}(0)\\&=-i(Sf-\hat{f}(0))\\&=Hf+i\hat{f}(0).\end{aligned}$$

In particular if $\hat{f}(0)=0$, then $\tilde{f}=Hf$. The weighted L^2-norm is defined by

$$\|f\|_w=\|f\|_{L^2(wdm)}=\left(\int_{\mathbf{T}}|f|^2wdm\right)^{1/2}.$$

Let $a,b\in L^\infty(\mathbf{T})=L^\infty(\mathbf{T},dm)$. If aP_++bP_- is a bounded operator on $L^2(wdm)$, then the operator norm of aP_++bP_- is defined by

$$\|aP_++bP_-\|=\|aP_++bP_-\|_{B(L^2(w))}=\sup\{\|(aP_++bP_-)f\|_w\,;\,f\in L^2(w),\ \|f\|_w=1\}.$$

H. Helson and G. Szegő proved that the Riesz projection P_+ is bounded on $L^2(d\mu)$ if and only if $d\mu=wdm$ is an (HS) measure (cf. [3], [12], [14], [15], [20], [21], [26]). Let H^1 denote the Hardy space. H^1 is a norm-closed subspace of $L^1(\mathbf{T})$. There is a deep extension of the Helson-Szegő approach developed in a series of papers by M. Cotlar, C. Sadosky, R. Arocena and M. Dominguez (cf. [21] Vol.1, p.132, [24], [4], [2], [1], [23], [5], [8], [6], [25], [7]). P. Koosis [16] established the two weights norm inequality for the Hilbert transform on weighted L^2 spaces. In particular, M. Cotlar and C. Sadosky proved the following:

Theorem A. (Cotlar-Sadosky) *For a positive constant M satisfying $M\geq 1$, the following are equivalent:*

(*i*) *the Hilbert transform H is bounded on $L^2(wdm)$ and $\|H\|\leq M$;*

(*ii*) *there exists $h\in H^1$ such that $|w-h|\leq\dfrac{M^2-1}{M^2+1}w=\left(1-\dfrac{2}{M^2+1}\right)w$ a.e. on $\mathbf{T}$;*

(*iii*) *there exists* $h \in H^1$ *such that* $|w-h|^2 \leq \left(1-\left(\frac{2M}{M^2+1}\right)^2\right)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w=\exp(u+\tilde{v}+const.)$,

$$\|v\|_\infty \leq \arccos\left(\frac{2M}{M^2+1}\right)=\frac{\pi}{2}-\arcsin\left(\frac{2M}{M^2+1}\right) \text{ and } |u| \leq \cosh^{-1}\left(\frac{M^2+1}{2M}\cos v\right) \text{ a.e. on } \mathbf{T},$$

where $\cosh^{-1}x=\log(x+\sqrt{x^2-1})$.

If the SIO $(aP_+ + bP_-)$ is bounded and invertible on the weighted space $L^2(wdm)$, then we shall consider the explicit form of the (HS) type weights w using functions arccos, arcsin and $\cosh^{-1}$ (cf. [19], [28], [29], [30]). In this paper, Theorems 3 and 4 are the main theorems.

2. MAIN THEOREMS AND COROLLARIES

In this section, we shall give the main theorems and their corollaries. We use Lemmas 1 and 2 to prove the main theorems. The proof of Lemma 1 is essential (cf. [19], [28], [29], [30]), which uses the Cotlar-Sadosky lifting theorem. There are many kinds of proofs of the lifting theorem.

Lemma 1. *For* $a,b \in L^\infty$ *and a positive constant* M *satisfying* $\max(|a|,|b|) \leq M$, *let* $d=\left|\frac{(a-b)M}{M^2-a\bar{b}}\right|$, *then* $\|d\|_\infty \leq 1$, *and the following are equivalent:*

(*i*) *the SIO* $(aP_+ + bP_-)$ *is bounded on* $L^2(wdm)$ *and* $\|aP_+ + bP_-\| \leq M$;

(*ii*) *there exists* $h \in H^1$ *such that* $|(M^2-a\bar{b})w-h|^2 \leq (M^2-|a|^2)(M^2-|b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iii*) *there exists* $h \in H^1$ *such that* $|w-h|^2 \leq (1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w=\exp(u+\tilde{v}+const.)$ *a.e.*,

$$|v| \leq \arccos d=\frac{\pi}{2}-\arcsin d \text{ a.e. and } |u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right) \text{ a.e. on } \mathbf{T}.$$

The proof of Lemma 1 is similar to the proof of Theorem A, and the proof of the following Lemma 2 is similar to one of Lemma 1.

Lemma 2. *For* $a,b \in L^\infty$ *and a positive constant* N *satisfying* $\min(|a|,|b|) \geq N$, *let* $d=\left|\frac{(a-b)N}{N^2-a\bar{b}}\right|$, *then* $\|d\|_\infty \leq 1$, *and the following are equivalent:*

(*i*) $N\|f\|_w \leq \|(aP_+ + bP_-)f\|_w$ *for every* $f \in \mathcal{P}$;

(*ii*) *there exists* $h \in H^1$ *such that* $|(N^2-a\bar{b})w-h|^2 \leq (N^2-|a|^2)(N^2-|b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iii*) *there exists* $h \in H^1$ *such that* $|w-h|^2 \leq (1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w = \exp(u + \tilde{v} + const.)$ *a.e.*,

$$|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d \text{ a.e. and } |u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right) \text{ a.e. on } \mathbf{T}.$$

The following Theorems 3 and 4 are the main theorems of this paper. These follow from Lemmas 1 and 2.

Theorem 3. *For* $a, b \in L^\infty$ *and positive constants* M *and* N *satisfying* $|M^2 - a\bar{b}| \cdot |N^2 - a\bar{b}| > 0$ *and* $N \leq \min(|a|, |b|) \leq \max(|a|, |b|) \leq M$, *let* $d = \max\left(\left|\frac{(a-b)M}{M^2 - a\bar{b}}\right|, \left|\frac{(a-b)N}{N^2 - a\bar{b}}\right|\right)$. *Then* $\|d\|_\infty \leq 1$, *and the following are equivalent:*

(*i*) $N\|f\|_w \leq \|(aP_+ + bP_-)f\|_w \leq M\|f\|_w$ *for every* $f \in \mathcal{P}$;

(*ii*) *there exists* $h \in H^1$ *such that* $|(M^2 - a\bar{b})w - h|^2 \leq (M^2 - |a|^2)(M^2 - |b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iii*) *there exists* $h \in H^1$ *such that* $|w-h|^2 \leq (1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w = \exp(u + \tilde{v} + const.)$ *a.e.*,

$$|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d \text{ a.e. and } |u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right) \text{ a.e. on } \mathbf{T}.$$

Theorem 4 follows from Theorem 3. If wdm is the (HS) measure, then there are many articles concerning the invertibility of the SIO $(aP_+ + bP_-)$ on $L^2(wdm)$ (cf. [3], [9], [10], [13], [17], [21], [22]). In Theorems 3 and 4, we do not assume that wdm is the (HS) measure. In Theorem 4, we establish the (HS) type representation of the weight function w using the operator norms of $aP_+ + bP_-$ and $(aP_+ + bP_-)^{-1}$.

Theorem 4. *For* $a, b \in L^\infty$ *such that the SIO* $(aP_+ + bP_-)$ *is bounded and invertible on* $L^2(wdm)$ *and* $\left|\|aP_+ + bP_-\|^2 - a\bar{b}\right| \cdot \left|\frac{1}{\|(aP_+ + bP_-)^{-1}\|^2} - a\bar{b}\right| > 0$ *a.e. on* $\mathbf{T}$, *let*

$$d = \max\left(\left|\frac{(a-b)\|aP_+ + bP_-\|}{\|aP_+ + bP_-\|^2 - a\bar{b}}\right|, \left|\frac{(a-b)\|(aP_+ + bP_-)^{-1}\|}{1 - a\bar{b}\|(aP_+ + bP_-)^{-1}\|^2}\right|\right).$$

Then $\|d\|_\infty \leq 1$, *and there exist real functions* u *and* v *such that* $w = \exp(u + \tilde{v} + const.)$ *a.e.*, $|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d$ *a.e. and* $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ *a.e. on* $\mathbf{T}$.

The following Corollaries 5, 6 and 7 follow from Theorem 3.

Corollary 5. *For distinct and non-zero complex constants a, b and positive constants M, N satisfying* $|M^2-a\bar{b}|\cdot|N^2-a\bar{b}|>0$ *and* $N\leq\min(|a|,|b|)\leq\max(|a|,|b|)\leq M$, *let* $d=\max\left(\left|\frac{(a-b)M}{M^2-a\bar{b}}\right|,\left|\frac{(a-b)N}{N^2-a\bar{b}}\right|\right)$, *then* $d\leq 1$, *and the following are equivalent:*

(i) $N\|f\|_w\leq\|(aP_++bP_-)f\|_w\leq M\|f\|_w$ *for every* $f\in\mathcal{P}$;

(ii) *there exist* $h, k\in H^1$ *such that* $|(M^2-a\bar{b})w-h|^2\leq(M^2-|a|^2)(M^2-|b|^2)w^2$ *a.e., and* $|(N^2-a\bar{b})w-k|^2\leq(N^2-|a|^2)(N^2-|b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(iii) *there exists* $h\in H^1$ *such that* $|w-h|^2\leq(1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(iv) *there exist real functions* u *and* v *such that* $w=\exp(u+\tilde{v}+const.)$ *a.e.,* $\|v\|_\infty\leq\arccos d=\frac{\pi}{2}-\arcsin d$ *and* $|u|\leq\cosh^{-1}\left(\frac{\cos v}{d}\right)$ *a.e. on* $\mathbf{T}$.

Corollary 6. *For distinct and non-zero complex constants a, b and a positive constant M satisfying* $\max(|a|,|b|)\leq M$, *let* $d=\left|\frac{(a-b)M}{M^2-a\bar{b}}\right|$, *then* $d\leq 1$, *and the following are equivalent:*

(i) *the SIO* (aP_++bP_-) *is bounded on* $L^2(wdm)$ *and* $\|aP_++bP_-\|\leq M$;

(ii) *the SIO* $(aP_++bP_-)^{-1}=a^{-1}P_++b^{-1}P_-$ *is bounded on* $L^2(wdm)$ *and* $\|(aP_++bP_-)^{-1}\|\leq\frac{M}{|ab|}$;

(iii) *there exists* $h\in H^1$ *such that* $|(M^2-a\bar{b})w-h|^2\leq(M^2-|a|^2)(M^2-|b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(iv) *there exists* $h\in H^1$ *such that* $|w-h|^2\leq(1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(v) *there exist real functions* $u, v\in L^\infty$ *such that* $w=\exp(u+\tilde{v}+const.)$ *a.e.,* $\|v\|_\infty\leq\arccos d=\frac{\pi}{2}-\arcsin d$ *and* $|u|\leq\cosh^{-1}\left(\frac{\cos v}{d}\right)$ *a.e. on* $\mathbf{T}$.

Corollary 7. *For positive constants M and N satisfying* $N\leq 1\leq M$, *let* $\gamma=\min(M,N^{-1})$, $\delta=\max\left(\frac{2}{M^2+1},\frac{2N^2}{N^2+1}\right)$ *and* $d=\max\left(\frac{2M}{M^2+1},\frac{2N}{N^2+1}\right)$. *Then* $\delta=\frac{2}{\gamma^2+1}$, $d=\frac{2\gamma}{\gamma^2+1}$, *and the following are equivalent:*

(i) $N\|f\|_w\leq\|Hf\|_w\leq M\|f\|_w$ *for every* $f\in\mathcal{P}$;

(ii) *there exists* $h\in H^1$ *such that* $|w-h|\leq(1-\delta)w$ *a.e. on* $\mathbf{T}$;

(iii) *there exists* $h\in H^1$ *such that* $|w-h|^2\leq(1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(iv) *there exist real functions* $u, v\in L^\infty$ *such that* $w=\exp(u+\tilde{v}+const.)$, $\|v\|_\infty\leq\arccos\left(\frac{2\gamma}{\gamma^2+1}\right)=\frac{\pi}{2}-\arcsin\left(\frac{2\gamma}{\gamma^2+1}\right)$ *and* $|u|\leq\cosh^{-1}\left(\frac{\gamma^2+1}{2\gamma}\cos v\right)$ *a.e. on* $\mathbf{T}$.

For distinct and non-zero complex constants a and b, the SIO $(aP_+ + bP_-)$ is bounded on $L^2(wdm)$ if and only if $(aP_+ + bP_-)$ is invertible on $L^2(wdm)$ if and only if the Riesz projection P_+ is bounded on $L^2(wdm)$. Then

$$\|(aP_+ + bP_-)^{-1}\| = \|a^{-1}P_+ + b^{-1}P_-\| = \frac{1}{|ab|}\|bP_+ + aP_-\| = \frac{1}{|ab|}\|aP_+ + bP_-\|.$$

By Theorem 4, we have:

Corollary 8. *For distinct and non-zero complex constants a and b, the SIO $(aP_+ + bP_-)$ is bounded on $L^2(wdm)$ if and only if $(aP_+ + bP_-)$ is invertible. Let*

$$d = \left|\frac{(a-b)\|aP_+ + bP_-\|}{\|aP_+ + bP_-\|^2 - a\bar{b}}\right|.$$

Then $0 < d \leq 1$, and there exist real functions u and v such that $w = \exp(u + \tilde{v} + const.)$ a.e., $\|v\|_\infty \leq \arccos d = \frac{\pi}{2} - \arcsin d$ and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ a.e. on $\mathbf{T}$.

Example. The relation between the norms of the operators H, P_+, P_- on the space $L^2(w)$;

$$\|P_+\| = \|P_-\| = \frac{\|H\| + \|H\|^{-1}}{2}$$

was remarked by Spitkovsky [27]. Then

$$\|H\| = \|S\| = \|P_+ - P_-\| = \|P_+\| + \sqrt{\|P_+\|^2 - 1}.$$

For $\zeta_0 \in \mathbf{T}$, and $-1 < \delta < 1$, let $w(\zeta) = |\zeta - \zeta_0|^\delta$. Then the equality $\|H\| = \cot\frac{\pi(1-|\delta|)}{4}$ was obtained by Krupnik and Verbitsky [18]. Hence $\|P_+\| = \sec\frac{\pi\delta}{2}$.

For complex constants a, b, the formula of the operator norm of $aP_+ + bP_-$ on $L^2(wdm)$ was obtained by Feldman, Krupnik and Markus (cf. [11], [13, Section 13.5], [30]) as the following:

$$\|aP_+ + bP_-\| = \sqrt{\gamma + \left(\frac{|a|+|b|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|a|-|b|}{2}\right)^2},$$

where

$$\gamma = \left|\frac{a-b}{2}\right|^2 (\|P_+\|^2 - 1).$$

If $\gamma = 0$, then the right hand side equals to $\max(|a|, |b|)$.

For example, suppose $w(\zeta)=|\zeta-1|^{1/2}$. Then $\|P_+\|=\sec\frac{\pi}{4}=\sqrt{2}$, and $\gamma=\left|\frac{a-b}{2}\right|^2$. Let

$$c=\|aP_+ + bP_-\|=\sqrt{\left|\frac{a-b}{2}\right|^2+\left(\frac{|a|+|b|}{2}\right)^2}+\sqrt{\left|\frac{a-b}{2}\right|^2+\left(\frac{|a|-|b|}{2}\right)^2},$$

and let $d=\left|\frac{(a-b)c}{c^2-a\bar{b}}\right|$. Then $d\leq 1$, and for every distinct complex constants a and b, we have $d=\frac{1}{\sqrt{2}}$ by the calculation, and $w(\zeta)=|\zeta-1|^{1/2}$ has the Helson-Szegő representation:

there exist real functions u and v such that $w=\exp(u+\tilde{v}+const.)$ a.e., $\|v\|_\infty\leq\arccos d=\frac{\pi}{4}$ and $|u|\leq\cosh^{-1}\left(\frac{\cos v}{d}\right)=\cosh^{-1}(\sqrt{2}\cos v)$ a.e. on $\mathbf{T}$.

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