タイトル	On Some Singular Integral Operators Which are One to One Mappings on the Weighted Lebesgue-Hilbert Spaces
著者	YAMAMOTO, Takanori
引用	北海学園大学学園論集(171): 11-24
発行日	2017-03-25

On Some Singular Integral Operators Which are One to One Mappings on the Weighted Lebesgue-Hilbert Spaces

Takanori YAMAMOTO

Dedicated to Professor Takahiko Nakazi on the occasion of his 70th birthday

ABSTRACT

Let ϕ be a bounded measurable function on the unit circle. Then we shall give the form of a weight W for which the singular integral operator $\phi P_+ + P_-$ is left invertible in the weighted space $L^2(W)$. P_+ is an analytic projection, P_- is a co-analytic projection. When W is an (A_2) weight, $\phi P_+ + P_-$ is left invertible (resp. invertible) in $L^2(W)$ if and only if Toeplitz operator T_ϕ is left invertible (resp. invertible) in $H^2(W)$.

KEYWORDS: Singular integral operator, Riesz projection, Hardy space
MSC (2010): Primary 46J15, 47B35.

§ 1. INTRODUCTION.

Let m denote the normalized Lebesgue measure on the unit circle $\mathbf{T}=\{\zeta\,;|\zeta|=1\}$ and let χ denote the identity function on $\mathbf{T}$. For a function f in $L^1(m)$, its k-th Fourier coefficient $\hat{f}(k)$ is defined by

$$\hat{f}(k)=\int_{\mathbf{T}}\chi^{-k} f\, dm$$

for all integers k. For a function f in $L^1(m)$, its harmonic conjugate function $\tilde{f}$ is defined by the singular integral

$$\tilde{f}(\theta)= VP\int_{\mathbf{T}} f(\theta-t)\cot\frac{t}{2}dm(t).$$

Let $\mathrm{C}(\mathbf{T})$ be an algebra of all continuous functions f on $\mathbf{T}$, and let A be a disc algebra of all functions f in $\mathrm{C}(\mathbf{T})$ such that $\hat{f}(k)=0$ for all negative integers k. The Hardy spaces H^p, $0<p\leqq\infty$, are defined as follows. For $0<p<\infty$, H^p is the $L^p(m)$-closure of A, while H^∞ is defined to be the weak-* closure of A in $L^\infty(m)$. If an f in H^p has the form $f=\exp(u+i\tilde{u}+ic)$ a.e. for some function u in $L^1_R(m)$ and some real constant c, then f is called an outer function. Let A_0 be the subspace of all functions f in A which satisfy $\hat{f}(0)=0$, and let $\bar{A}_0$ be the subspace of all complex conjugate functions of functions in A_0. Since the intersection of H^1 and $\bar{H}^1_0$ is only the zero function, the analytic projection P_+ is defined as

$$P_+(f_1+f_2)=f_1, \text{ for all } f_1 \text{ in } H^1 \text{ and all } f_2 \text{ in } \bar{H}^1_0.$$

The co-analytic projection P_- is defined by $P_-=I-P_+$ where I is an identity operator on $H^1+\bar{H}^1_0$. Then

$$P_\pm f=\frac{1}{2}\{f\pm i\tilde{f}\pm\hat{f}(0)\}, \text{ for all } f \text{ in } A+\bar{A}_0.$$

For a ϕ in $L^\infty(m)$, the Toeplitz operator T_ϕ is defined as a map from H^2 to H^2 by

$$T_\phi f=P_+(\phi f), \text{ for all } f \text{ in } H^2.$$

A non-negative integrable function W on $\mathbf{T}$ is said to be a weight. P_+ is bounded on $L^p(W)$ if and only if W satisfies the A_p-condition (cf.[6], p.254). (A_p) denotes the set of all positive weights W satisfying the A_p-condition. In the case $p=2$, Helson-Szegö theorem gives the form of a weight W in (A_2) (cf.[6], p.147 and [7]). If W is in (A_2), then T_ϕ is bounded in $H^2(W)$ and $\phi P_+ + P_-$ is bounded in $L^2(W)$. A weight W does not necessarily belong to (A_2) when those operators are bounded. In this paper we shall give the form of a weight W such that $\phi P_+ + P_-$ is bounded and left invertible in $L^2(W)$. It should be mentioned that W is in (A_2) if and only if there exist a function k in H^1 and a constant ρ, $\rho<1$ such that $|W-k|\leqq\rho W$ a.e.. If W is in (A_2), then $\log W$ is in $\mathrm{BMO}=L^\infty_R(m)+\tilde{L}^\infty_R(m)$.

Definition. (1) For a function λ in $L^\infty(m)$,

$$A(\lambda)=\{s\in\mathrm{BMO}\,;\,\lambda=|\lambda|\exp(is)\ a.e.\},$$
$$\Lambda=\{\lambda\in L^\infty(m)\,;\,|\lambda|\exp(\tilde{s}) \text{ is bounded for some } s \text{ in } A(\lambda)\}.$$

(2) For a function ϕ in $L^\infty(m)$, we shall wright

$$E(\phi)=\{\zeta\in\mathbf{T}\,;\,\phi(\zeta)\neq 1\}\ and\ m(E(\phi))=\int_E dm=\int_E\frac{dt}{2\pi}.$$

$I(\phi,+)$, $I(\phi,-)$ denote intervals such that

$$I(\phi,+)=[\max\{1,\|\phi\|_\infty\},\infty),$$
$$I(\phi,-)=(0,\min\{1,\text{ess inf }|\phi|\}] \text{ and put}$$

$$I(\phi)=I(\phi,+)\cup I(\phi,-),$$
$$J(\phi)=\{t\in I(\phi)\,;\,t^2-\phi \text{ belongs to } \Lambda\}.$$

(3) For a function ϕ in $L^\infty(m)$ and a constant t in $I(\phi)$ satisfying $m\{\phi=t^2\}=0$, put

$$r(t,\phi)=\left|\frac{(\phi-1)t}{t^2-\phi}\right|,$$

and for a function v satisfying $|v|\leqq\cos^{-1} r(t,\phi)$ a.e., put

$$U(t,\phi,v)=\cosh^{-1}\left(\frac{\cos v}{r(t,\phi)}\right).$$

In this paper we shall assume $-\pi\leqq\operatorname{Arg} z<\pi$. For any ϕ in $L^\infty(m)$, $0\leqq m(E(\phi))\leqq 1$. If $|\phi|=1$ a.e., then $I(\phi)=(0,\infty)$. For any λ in $L^\infty(m)$, $\operatorname{Arg}\lambda$ belongs to a set $A(\lambda)$. $\Lambda\cdot\Lambda=\Lambda$ and Λ contains a set $\exp H^\infty$. λ belongs to Λ if and only if there exist two functions t, s in $L^\infty_R(m)$ such that $t+\tilde{s}$ is bounded above and $\lambda=\exp(t+is)$ a.e.. The following Lemma A is useful to study the boundedness and the left invertibility of $\phi P_+ + P_-$ in $L^2(W)$.

Lemma A. *Suppose ϕ is a function in $L^\infty(m)$ such that $m(E(\phi))>0$. Suppose t is a constant in $J(\phi)$ such that $m\{\phi=t^2\}=0$. Then $r(t,\phi)\leqq 1$ a.e.. For a weight W such that* $\log W$ *is integrable, the following conditions are equivalent.*

(i) *There exists a function k in H^1 such that*

$$|(t^2-\phi)W-k|\leqq\{1-r(t,\phi)^2\}^{1/2}\,|t^2-\phi|W \text{ a.e.}.$$

(ii) *There exist three functions u, v, s, and a constant c such that*

$$|v|\leqq\cos^{-1} r(t,\phi) \text{ a.e., and } m\{|v|=\pi/2\}=0\,;$$

$$|u|\leqq U(t,\phi,v) \text{ a.e. on } E(\phi), \text{ and } -\log(2\cos v)\leqq u \text{ a.e. on } E(\phi)^c;$$

$$s \text{ is in } A(t^2-\phi), \text{ and } W=\left(\chi_{E(\phi)^c}\frac{t}{|t^2-1|}+\chi_{E(\phi)}\frac{1}{|\phi-1|}\right)\exp(u-\tilde{v}-\tilde{s}-c) \text{ a.e.}.$$

If $m(E(\phi)^c)>0$ then $t\neq 1$. If W satisfies one of these conditions, then W^{-1} is integrable.

For a given function ϕ in $L^\infty(m)$, the form of a weight W such that $\phi P_+ + P_-$ is bounded in $L^2(W)$ was given in our preceding paper [14]. The proof of Lemma A is similar to it. In § 2, we shall give the proof. It is known that T_ϕ is left invertible (resp. invertible) in H^2 if and only if $\phi P_+ + P_-$ is left invertible (resp. invertible) in $L^2(m)$ (cf.[10], p.71 and [15], p.393). Left invertibilities of singular integral operators $\phi P_+ + P_-$ and Toeplitz operators T_ϕ in weighted spaces were never been studied. In § 3, we shall give the form of a weight W such that $\phi P_+ + P_-$ (resp. T_ϕ) is bounded and left invertible in $L^2(W)$ (resp. $H^2(W)$). A central role is played by the Cotlar-Sadosky lifting theorem and Lemma A. The invertibility of T_ϕ in weighted spaces was already studied by Rochberg [16]. In §4, we shall consider the invertibility of $\phi P_+ + P_-$ and T_ϕ in weighted spaces. For a function f in $L^2(W)$, the $L^2(W)$ norm of f is denoted by $\|f\|_W = \left\{\int_{\mathbf{T}} |f|^2 \, W dm\right\}^{1/2}$.

§2. PROOF OF LEMMA A.

We shall show that (*i*) implies (*ii*). Suppose $k=0$ in (*i*), then by the calculation we have $\phi=1$ a.e. which contradicts to $m(E(\phi))>0$. Hence we have $k\neq 0$. Since t is in $I(\phi)$, we have

$$|t^2-\phi|^2-|\phi-1|^2 t^2=(t^2-1)(t^2-|\phi|^2)\geqq 0 \text{ a.e..}$$

Hence $r(t,\phi)\leqq 1$ a.e. and $|k|\leqq 2W|t^2-\phi|$ a.e.. Suppose $m(E(\phi)^c)>0$ and $t=1$ in (*i*), then $k=0$ a.e. on $E(\phi)^c$ and hence $k=0$ a.e.. This contradiction implies that if $m(E(\phi)^c)>0$ then $t\neq 1$. Since t is in $J(\phi)$, $t^2-\phi$ belongs to Λ. Hence there exists a function s in $L^\infty_R(m)$ such that $t^2-\phi = |t^2-\phi| \exp(is)$ a.e. and $|t^2-\phi|\exp(\tilde{s})$ is bounded. Put $g=k\exp(\tilde{s}-is)$, then $|g|\leqq 2W|t^2-\phi|\exp(\tilde{s})$ a.e.. Hence g is a non-zero function in H^1. Put $v=\operatorname{Arg} g$, then $|v|\leqq \cos^{-1} r(t,\phi)$ a.e. since

$$v=\operatorname{Arg}\left(\frac{k}{t^2-\phi}\right) \text{ a.e., and } \left|W-\frac{k}{t^2-\phi}\right|\leqq\left\{1-r(t,\phi)^2\right\}^{1/2} W \text{ a.e..}$$

Since g is an outer function such that Re $g\geqq 0$ a.e. and

$$\frac{\exp(iv-\tilde{v})}{|\exp(iv-\tilde{v})|}=\frac{g}{|g|} \text{ a.e.,}$$

there exists a positive constant γ such that $\exp(iv-\tilde{v})=\gamma g$ a.e. (cf.[11], Proposition 5). Put

$$u=\tilde{v}+\tilde{s}+\log W+\log t+\log\gamma+\chi_{E(\phi)}\log|\phi-1|+\chi_{E(\phi)^c}\log|t-t^{-1}|,$$

then

$$W=\left(\chi_{E(\phi)^c}\frac{t}{|t^2-1|}+\chi_{E(\phi)}\frac{1}{|\phi-1|}\right)\exp(u-\tilde{v}-\tilde{s}-c) \text{ a.e..}$$

Since $|1-r(t,\phi)\exp(iv-u)|^2\leqq 1-r(t,\phi)^2$ a.e. on $E(\phi)$, we have

$$e^{2u}-2\left(\frac{\cos v}{r(t,\phi)}\right)e^u+1\leqq 0 \text{ a.e. on } E(\phi),$$

and hence $|u|\leqq U(t,\phi,v)$ a.e. on $E(\phi)$. Since

$$\left|\frac{k}{t^2-\phi}\right|^2\leqq 2W\operatorname{Re}\left(\frac{k}{t^2-\phi}\right) \text{ a.e.,}$$

we have

$$\left|\frac{k}{t^2-\phi}\right| \leqq 2W \cos v \text{ a.e..}$$

Hence $W^{-1} \leqq 2\gamma|t^2-\phi|\exp(\tilde{v}+\tilde{s}) \cos v$ a.e.. Since $|v| \leqq \pi/2$ a.e., $\exp(\tilde{v})\cos v$ is integrable (cf.[6], p. 161). Since t is in $J(\phi)$, W^{-1} is integrable. Since $(\cos v)^{-p}$ is integrable for some p, $p>0$, we have $m\{|v|=\pi/2\}=0$. Since

$$2\gamma W|t^2-1|\exp(\tilde{v}+\tilde{s}) \cos v \geqq 1 \text{ a.e. on } E(\phi)^c,$$

we have $-\log(2 \cos v) \leqq u$ a.e. on $E(\phi)^c$. We shall show that (*ii*) implies (*i*). Since $|u| \leqq U(t, \phi, v)$ a.e. on $E(\phi)$, we have

$$|1-r(t,\phi)\exp(iv-u)|^2-\{1-r(t,\phi)^2\}=r(t,\phi)^2\left\{e^{-2u}-2\left(\frac{\cos v}{r(t,\phi)}\right)e^{-u}+1\right\} \leqq 0 \text{ a.e. on } E(\phi).$$

Put $k=t\exp\{i(v+s)-(v+s)^{\sim}-c\}$, then

$$|(t^2-\phi)W-k|=|1-r(t,\phi)\exp(iv-u)|\cdot|t^2-\phi|W \leqq \{1-r(t,\phi)^2\}^{1/2}|t^2-\phi|W \text{ a.e. on } E(\phi).$$

Since $-\log(2 \cos v) \leqq u$ a.e. on $E(\phi)^c$, we have $|1-\exp(iv-u)| \leqq 1$ a.e. on $E(\phi)^c$. Hence

$$|(t^2-1)W-k|=|t^2-1|\cdot|1-\exp(iv-u)|W \leqq |t^2-1|W \text{ a.e. on } E(\phi)^c.$$

Since $|k| \leqq 2|t^2-\phi|W$ a.e., k is in H^1. Hence (*i*) follows. This completes the proof.

If $\chi_{E(\phi)}\log|\phi-1|$ is integrable, then it is possible to take an integrable function u in condition (*ii*). If $r(t,\phi)$ is bounded away from zero, then it is possible to take a bounded function u in (*ii*).

§ 3. LEFT INVERTIBILITY.

We shall give the form of a weight W such that $\phi P_+ + P_-$ is bounded and left invertible in $L^2(W)$. If W is in (A_2), then $\phi P_+ + P_-$ is left invertible in $L^2(W)$ if and only if T_ϕ is left invertible in $H^2(W)$.

Definition. For a t in $I(\phi)$ and a ϕ in $L^\infty(m)$, let

$$\begin{aligned} L(t,\phi)=\{\ell=u-\tilde{v}-\tilde{s}-c\,;\ & \\ & |v| \leqq \cos^{-1} r(t,\phi) \text{ a.e., } m\{v=\pi/2\}=0. \\ & |u| \leqq U(t,\phi,v) \text{ a.e. on } E(\phi), \text{ and } -\log(2\cos v) \leqq u \text{ a.e. on } E(\phi)^c. \\ & s \in A(t^2-\phi), \text{ and } c \text{ is a real constant.}\}. \end{aligned}$$

If $r(t, \phi)$ is bounded away from zero, then $L(t, \phi)$ is a convex subset of BMO.

Theorem 1. *Suppose ϕ is a function in $L^\infty(m)$ such that $m(E(\phi))>0$. Suppose ε is a positive constant such that both ε and ε^{-1} belong to $J(\phi)$. For a weight W such that* log W *is integrable, the following conditions are equivalent.*

(*i*) $\varepsilon\|f\|_W \leqq \|(\phi P_+ + P_-)f\|_W \leqq \varepsilon^{-1}\|f\|_W$, *for all f in $A+\bar{A}_0$.*

(*ii*) $\varepsilon \leqq 1$, $\varepsilon \leqq |\phi| \leqq \varepsilon^{-1}$ *a.e.*, $m\{\phi=\varepsilon^2\}=m\{\phi=\varepsilon^{-2}\}=0$ *and there exists an ℓ in $L(\varepsilon, \phi)\cap L(\varepsilon^{-1}, \phi)$ such that*

$$W=\left(\frac{\varepsilon}{|\varepsilon^2-1|}\chi_{E(\phi)^c}+\frac{1}{|\phi-1|}\chi_{E(\phi)}\right)\exp \ell \text{ a.e.}.$$

If $m(E(\phi)^c)>0$ then $\varepsilon\neq 1$. If W satisfies one of these conditions, then W^{-1} is integrable.

Proof. By Cotlar-Sadosky's theorem [4], if follows from (*i*) that there exist two functions k, k' in H^1 such that

$$|(\varepsilon^2-\phi)W-k|^2 \leqq (\varepsilon^2-1)(\varepsilon^2-|\phi|^2)W^2 \text{ a.e.},$$
$$|(\varepsilon^{-2}-\phi)W-k'|^2 \leqq (\varepsilon^{-2}-1)(\varepsilon^{-2}-|\phi|^2)W^2 \text{ a.e.}.$$

Since $m(E(\phi))>0$, it follows that k and k' are non-zero functions. Suppose $m\{\phi=\varepsilon^2\}>0$, then $m\{k=0\}>0$. Since k is in H^1, we have $k=0$ a.e. (cf.[8], p.76). This contradiction implies $m\{\phi=\varepsilon^2\}=0$. In the same way we have $m\{\phi=\varepsilon^{-2}\}=0$. Then

$$(\varepsilon^{\pm 2}-1)(\varepsilon^{\pm 2}-|\phi|^2)=\{1-r(\varepsilon^{\pm 1}, \phi)^2\}|t^2-\phi|^2 \text{ a.e.}.$$

We use Lemma A to complete the proof.

Remark 1. For a function ϕ such that $|\phi|=1$ a.e., we have $r(\varepsilon, \phi)=r(\varepsilon^{-1}, \phi)$ a.e. and hence $U(\varepsilon, \phi, v)=U(\varepsilon^{-1}, \phi, v)$ a.e.. In this case the condition (*ii*) in the above theorem becomes as follows.
(*ii*)′ There exist three functions u, v, s and a constant c such that

$$W=\left(\frac{\varepsilon}{|\varepsilon^2-1|}\chi_{E(\phi)^c}+\frac{1}{|\phi-1|}\chi_{E(\phi)}\right)\exp(u-\tilde{v}-\tilde{s}-c) \text{ a.e.},$$

where $|v| \leqq \cos^{-1} r(\varepsilon, \phi)$ a.e., $m\{|v|=\pi/2\}=0$;
$|u| \leqq U(\varepsilon, \phi, v)$ a.e. on $E(\phi)$, and
$-\log(2\cos v) \leqq u$ a.e. on $E(\phi)^c$; $s\in A(\varepsilon^2-\phi)\cap A(\varepsilon^{-2}-\phi)$.

It should be mentioned that if $\phi=-1$ a.e., then the condition (*ii*)′ becomes the Arocena, Cotlar and Sadosky's condition (cf.[1], [3] and [4]). In this case $\phi P_+ + P_- = -P_+ + P_-$ is invertible if and only if it is bounded. Then $E(-1)=\mathbf{T}$, $r(\varepsilon, -1) = r(\varepsilon^{-1}, -1) = 2\varepsilon/(1+\varepsilon^2)$ a.e., and $A(\varepsilon^2+1) \cap A(\varepsilon^{-2}+1)$ contains a function $s=0$.

Corollary 1. Suppose ϕ is a function in $L^\infty(m)$ such that $|\phi-1|>0$ a.e. and $J(\phi)$ contains a constant 1. For a weight W such that $\log W$ is integrable, the following conditions are equivalent.

(*i*) $\phi P_+ + P_-$ is an isometry in $L^2(W)$.

(*ii*) $|\phi|=1$ a.e., and there exist an s in $A(1-\phi)$ and a positive constant C such that

$$W=\frac{C}{|\phi-1|}\exp(-\tilde{s}) \text{ a.e..}$$

If W satisfies one of these conditions, then W^{-1} is bounded.

Proof. It follows from (*i*) that

$$\|(\phi P_+ + P_-)f\|_W = \|f\|_W, \text{ for all } f \text{ in } A+\bar{A}_0.$$

This is the case $\varepsilon=1$ in Theorem 1. Hence, $|\phi|=1$ a.e. and there exists an ℓ in $L(1,\phi)$ such that $W=|\phi-1|^{-1}\exp\ell$ a.e.. Since $r(1,\phi)=1$ a.e., we have

$$L(1,\phi)=\{-\tilde{s}-c\,;\, s\in A(1-\phi), \text{ and } c \text{ is a real constant}\}.$$

Since $J(\phi)$ contains 1, $|1-\phi|\exp(\tilde{s})$ is bounded for some s in $A(1-\phi)$ and hence W^{-1} is bounded. We use Theorem 1 to complete the proof.

Definition. For a function ϕ in $L^\infty(m)$, let $L(\phi,+)$, $L(\phi,-)$ and $L(\phi)$ denote subsets of real measurable functions such that

$$L(\phi,\pm)=\bigcup_{t\in I(\phi,\pm)} L(t,\phi) \text{ and } L(\phi)=L(\phi,+)\cap L(\phi,-).$$

Theorem 2. *Suppose ϕ is a function in $L^\infty(m)$ such that $|\phi-1|>0$ a.e.. Suppose there exists a positive constant δ such that $(0,\delta]\cup[\delta^{-1},\infty)$ is a subset of $J(\phi)$. For a weight W such that* $\log W$ *is integrable, the following conditions are equivalent.*

(*i*) *$\phi P_+ + P_-$ is bounded and left invertible in $L^2(W)$.*

(*ii*) *ϕ is bounded away from zero and there exists a function ℓ in $L(\phi)$* such that $W=|\phi-1|^{-1}$ $\exp\ell$ *a.e..*

If W satisfies one of these conditions, then W^{-1} is integrable.

Proof. We shall show that (*i*) implies (*ii*). By (*i*), there exists a positive constant ε such that both ε and ε^{-1} belong to $J(\phi)$ and

$$\varepsilon\|f\|_W \leqq \|(\phi P_+ + P_-)f\|_W \leqq \varepsilon^{-1}\|f\|_W, \text{ for all } f \text{ in } A+\bar{A}_0.$$

By Theorem 1, there exists an ℓ in $L(\varepsilon, \phi)\cap L(\varepsilon^{-1}, \phi)$ such that $W=|\phi-1|^{-1}\exp\ell$ a.e.. Since $L(\varepsilon, \phi)\cap L(\varepsilon^{-1}, \phi)$ is a subset of $L(\phi)$, (*ii*) follows. The converse is also true. This completes the proof.

Proposition 3. *Suppose* $|\phi-1|>0$ *a.e.. Let* t *and* t' *be positive constants satisfying* $t<t'$. *If* $J(\phi)=I(\phi)$, *then the following statements are true.*

(1) If t and t' belong to $I(\phi, +)$, then $L(t, \phi)$ is a subset of $L(t', \phi)$ and $r(t', \phi)\leqq r(t, \phi)$ a.e..

(2) If t and t' belong to $I(\phi, -)$, then $L(t', \phi)$ is a subset of $L(t, \phi)$ and $r(t, \phi)\leqq r(t', \phi)$ a.e..

Proof. Put $r=r(t, \phi)$ and $r'=r(t', \phi)$, then

$$r'^2-r^2=\frac{(t'^2-t^2)(|\phi|^2-t'^2t^2)}{|t'^2-\phi|^2|t^2-\phi|^2} \quad a.e..$$

We shall prove (1). Since t and t' belong to $I(\phi, +)$, we have $r'\leqq r$ a.e.. Let ℓ be in $L(t, \phi)$ and put $W=|\phi-1|^{-1}\exp\ell$, then it follows from Lemma A that there exists a k in H^1 such that

$$|(t^2-\phi)W-k|^2\leqq(t^2-1)(t^2-|\phi|^2)W^2 \ a.e..$$

By Cotlar-Sadosky's theorem [4],

$$\|(\phi P_+ + P_-)f\|_W\leqq t\|f\|_W\leqq t'\|f\|_W,$$

for all f in $A+\bar{A}_0$. By Cotlar-Sadosky's theorem, there exists a k' in H^1 such that

$$|(t'^2-\phi)W-k'|^2\leqq(t'^2-1)(t'^2-|\phi|^2)W^2 \ a.e..$$

By Lemma A, there exists an ℓ' in $L(t', \phi)$ such that $W=|\phi-1|^{-1}\exp\ell'$ a.e. and hence $\ell=\ell'$ a.e.. Thus $L(t, \phi)$ is a subset of $L(t', \phi)$. The proof of (2) is similar to one of (1). This completes the proof.

Proposition 4. *If* $J(\phi)=I(\phi)$ *and* $r(t, \phi)$ *is bounded away from zero for all* t *in* $I(\phi)$, *then* $L(\phi, +)$, $L(\phi, -)$ *and* $L(\phi)$ *are convex subsets of BMO.*

Proof. Let ℓ and ℓ' be in $L(\phi, +)$. There exist t and t' in $I(\phi, +)$ such that ℓ is in $L(t, \phi)$ and ℓ' is in $L(t', \phi)$. Since $r(t, \phi)$ is bounded away from zero, we have $|\phi-1|>0$ a.e. and $U(t, \phi, v)$ is in $L^\infty_R(m)$. Since $|\phi-1|>0$ a.e. and $J(\phi)=I(\phi)$, it follows from Proposition 3 that the convex combination of ℓ and ℓ' belongs to either $L(t, \phi)$ or $L(t', \phi)$ which is a convex subset of $L(\phi, +)$. Hence $L(\phi, +)$ is a convex subset of BMO. It follows in the similar way that $L(\phi, -)$ is convex and hence $L(\phi)$ is also convex.

Proposition 5. (1) If ϕ is an outer function in H^∞, then $J(\phi)\cup\{1\}=I(\phi)$.

(2) If ϕ is a function in $L^\infty_R(m)$ such that (ess inf ϕ, ess sup ϕ) does not contain zero, then $J(\phi)=I(\phi)$.

Proof. We shall prove (1). Let t be any constant in $I(\phi, +)$ not equal to one. Put $\lambda=t^2-\phi$, then λ is an invertible function in H^∞ since $|\lambda|\geqq t^2-\mathrm{Re}\ \phi\geqq t^2-\max\{t, 1\}>0$ a.e.. Hence there exist a function f and a constant c such that $\lambda=\exp(f+i\tilde{f}+ic)$ a.e.. Put $s=\tilde{f}+c$, then s is in $A(\lambda)$ since $|\lambda|\exp(\tilde{s})=c'$ for some constant c'. Thus $I(\phi, +)$ is a subset of $J(\phi)\cup\{1\}$. Let t be any constant in $I(\phi, -)$ not equal to one. We may assume that ϕ is bounded away from zero. Put $\lambda=t^2-\phi$, then λ is an invertible function in H^∞ since $\mathrm{Re}(t^{-1}-\phi^{-1})\geqq 0$ a.e. and $|\lambda|\geqq(1-t)(ess\ inf\,|\phi|)>0$ a.e.. Thus $I(\phi, -)$ is a subset of $J(\phi)\cup\{1\}$. Hence $J(\phi)\cup\{1\}=I(\phi)$. We shall prove (2). Let t be any constant in $I(\phi, +)$. Put $\lambda=t^2-\phi$, then λ is in $L^\infty_R(m)$ and $\lambda\geqq 0$ a.e. since $|\phi|\leqq t\leqq t^2$ a.e.. Put $s=\mathrm{Arg}\,\lambda$, then $s=0$ a.e. and hence $|\lambda|\exp(\tilde{s})$ is bounded. Thus $I(\phi, +)$ is a subset of $J(\phi)$. Let t be any constant in $I(\phi, -)$. Since (*ess inf* ϕ, *ess sup* ϕ) does not contain zero, we have $\phi\geqq 0$ a.e. or $\phi\leqq 0$ a.e.. If $\phi\geqq 0$ a.e., then $\lambda\leqq 0$ a.e. since $\phi\geqq t\geqq t^2$ a.e.. Put $s=\mathrm{Arg}\,\lambda$, then $s=-\pi$ a.e. and hence $|\lambda|\exp(\tilde{s})$ is bounded. Thus $I(\phi, -)$ is a subset of $J(\phi)$. If $\phi\leqq 0$ a.e., then $\lambda\geqq 0$ a.e. and hence $I(\phi, -)$ is a subset of $J(\phi)$. Hence $J(\phi)=I(\phi)$. This completes the proof.

For a weight W, $H^2(W)$ (*resp.* $H^2_0(W)$) denotes the $L^2(W)$-norm closure of A (*resp.* A_0). If W is in (A_2), then T_ϕ is bounded in $H^2(W)$ and $\phi P_+ + P_-$ is bounded in $L^2(W)$.

Proposition 6. *Let ϕ be a function in $L^\infty(m)$. For a W in (A_2), the following conditions are equivalent.*

(*i*) *$\phi P_+ + P_-$ is left invertible in $L^2(W)$.*

(*ii*) *$P_+\phi P_+ + P_-$ is left invertible in $L^2(W)$.*

(*iii*) *$T_\phi f$ is left invertible in $H^2(W)$.*

Proof. Put

$$\varepsilon_1=\inf\{\|(\phi P_+ + P_-)f\|_W\,;\,f\in A+\bar{A}_0,\ \|f\|_W=1\},$$
$$\varepsilon_2=\inf\{\|(P_+\phi P_+ + P_-)f\|_W\,;\,f\in A+\bar{A}_0,\ \|f\|_W=1\},\ \text{and}$$
$$\varepsilon_3=\inf\{\|T_\phi f\|_W\,;\,f\in A,\ \|f\|_W=1\}.$$

Suppose $\varepsilon_1>0$ and let f be any function in $A+\bar{A}_0$ satisfying $\|f\|_W=1$. Since $P_+\phi P_+ + P_- = (\phi P_+ + P_-)(I-P_-\phi P_+)$,

$$\|(P_+\phi P_+ + P_-)f\|_W \geqq \varepsilon_1 \|(I - P_-\phi P_+)f\|_W \geqq \varepsilon_1 \|I + P_-\phi P_+\|_W^{-1},$$

it follows that $\varepsilon_2 \geqq \varepsilon_1 \|I + P_-\phi P_+\|_W^{-1} > 0$. Hence (*i*) implies (*ii*). Suppose $\varepsilon_2 > 0$ and let f be any function in A satisfying $\|f\|_W = 1$. Since $\|T_\phi f\|_W \geqq \|(P_+\phi P_+ + P_-)f\|_W \geqq \varepsilon_2$, we have $\varepsilon_3 \geqq \varepsilon_2 > 0$. Hence (*ii*) implies (*iii*). Suppose $\varepsilon_3 > 0$ and let f be any function in $A + \bar{A}_0$ satisfying $\|f\|_W = 1$. Since $\|P_+\|_W = \|P_-\|_W$ (cf.[14]),

$$\varepsilon_3 \leqq \varepsilon_3(\|P_+ f\|_W + \|P_- f\|_W) \leqq \|T_\phi(P_+ f)\|_W + \varepsilon_3 \|P_- f\|_W \leqq (1+\varepsilon_3)\|P_+\|_W \|(P_+\phi P_+ + P_-)f\|_W.$$

We have $\varepsilon_3 \leqq \varepsilon_2(1+\varepsilon_3)\|P_+\|_W$ and hence $\varepsilon_2 > 0$. Hence (*iii*) implies (*ii*). Suppose $\varepsilon_2 > 0$ and let f be any function in $A + \bar{A}_0$ satisfying $\|f\|_W = 1$. Since $\phi P_+ + P_- = (P_+\phi P_+ + P_-)(I + P_-\phi P_+)$,

$$\|(\phi P_+ + P_-)f\|_W \geqq \varepsilon_2 \|(I + P_-\phi P_+)f\|_W \geqq \varepsilon_2 \|I - P_-\phi P_+\|_W^{-1},$$

we have $\varepsilon_1 \geqq \varepsilon_2 \|I - P_-\phi P_+\|_W^{-1} > 0$. Hence (*ii*) implies (*i*). This completes the proof.

Proposition 7. Suppose ϕ is a function in $L_R^\infty(m)$ such that $\phi - 1$ is bounded away from zero, and [ess inf ϕ, ess sup ϕ] does not contain zero. If $\phi P_+ + P_-$ is left invertible in $L^2(W)$, then W is in (A_2).

Proof. Since [ess inf ϕ, ess sup ϕ] does not contain zero and $\phi P_+ + P_-$ is left invertible, it follows that there exists a constant ε in $I(\phi)$ such that ε^2 does not belong to [ess inf ϕ, ess sup ϕ] and

$$\varepsilon \|f\|_W \leqq \|(\phi P_+ + P_-)f\|_W, \text{ for all } f \text{ in } A + \bar{A}_0.$$

By Cotlar-Sadosky's theorem, there exists a k in H^1 such that

$$|(\phi - \varepsilon^2)W - k| \leqq \{(|\phi|^2 - \varepsilon^2)(1-\varepsilon^2)\}^{1/2} W \leqq \{1 - r(\varepsilon, \phi)^2\}^{1/2} |\phi - \varepsilon^2| W \text{ a.e..}$$

Since $\phi - \varepsilon^2$ and $\phi - 1$ are bounded away from zero, it follows that $r(\varepsilon, \phi)$ is bounded away from zero. Then $\phi - \varepsilon^2 > 0$ a.e. or $\phi - \varepsilon^2 < 0$ a.e.. By Lemma A, $|\phi - \varepsilon^2| W$ is in (A_2) and hence W is in (A_2). This completes the proof.

Remark 2. *Suppose E is a Borel subset of a unit circle. Suppose ℓ is a function in $L_R^1(m)$ such that exp ℓ is integrable, not in (A_2), $-\log 2 \leqq \ell$ a.e. on E^c, and $|\ell| \leqq \cosh^{-1}\{(1+\varepsilon^2)/(2\varepsilon)\}$ a.e. on E. For a constant ε satisfying $0 < \varepsilon \leqq 1$, put*

$$R(E, \varepsilon) = \{W ; \varepsilon \|f\|_W \leqq \|((1 - 2\chi_E)P_+ + P_-)f\|_W \leqq \varepsilon^{-1} \|f\|_W, \text{ for all } f \text{ in } A + \bar{A}_0\}.$$

The following statements are then true.

(a) If $0<m(E)<1$, $0<\varepsilon<1$ and $W=\{(2\varepsilon)/(1-\varepsilon^2)\chi_{E^c}+\chi_E\}\exp\ell$, then W is in $R(E,\varepsilon)$, not in (A_2).

(b) If $m(E)=1$, then $(1-2\chi_E)P_++P_-=-P_++P_-$ and hence $R(E,\varepsilon)$ is a subset of (A_2).

In this section, we have assumed that $\log W$ is integrable. Similar results hold on the assumption that $W>0$ a.e.. If $m\{W=0\}>0$, then the following conditions are equivalent.

(i) ϕP_++P_- is bounded and left invertible in $L^2(W)$.

(ii) $W=0$ a.e. on $E(\phi)$, and W has no restriction on $E(\phi)^c$.

§ 4. INVERTIBILITY.

We shall consider the invertibility of operators ϕP_++P_- and T_ϕ in weighted spaces. If W is in (A_2), then ϕP_++P_- is invertible in $L^2(W)$ if and only if T_ϕ is invertible in $H^2(W)$. We shall use Rochberg theorem (cf.[16]) to prove Theorem 8.

Theorem 8. *Let ϕ be a function in $L^\infty(m)$. For a W in (A_2), the following conditions are equivalent.*

(i) *ϕP_++P_- is invertible in $L^2(W)$.*

(ii) *T_ϕ is invertible in $H^2(W)$.*

(iii) *ϕ can be written as*

$$\phi=\exp(U+ic-i\tilde{V})\ a.e.$$

with c a real constant; U a function in $L^\infty_R(m)$; V a real measurable function such that We^V is in (A_2).

If ϕ and W satisfy one of these conditions, then

$$\|I+P_-\phi P_+\|_W^{-1}\|T_\phi^{-1}\|_W\leqq\|(\phi P_++P_-)^{-1}\|_W\leqq(1+\|T_\phi^{-1}\|_W)\|P_+\|_W\|I-P_-\phi P_+\|_W.$$

Proof. Rochberg [16] proved (ii) is equivalent to (iii). We shall show that (i) implies (ii). By Proposition 6, (i) implies that T_ϕ is left invertible in $H^2(W)$. Let g be any function in $L^2(W)$. Since ϕP_++P_- has a dense range in $L^2(W)$, $T_\phi P_+=P_+(\phi P_++P_-)$ on $A+\bar{A}_0$, and P_+ is bounded in $L^2(W)$, it follows that T_ϕ has a dense range in $H^2(W)$. We shall show that (iii) implies (i) parallel to Rochberg [16]. Let ϕ_1 be a function such that

$$\phi_1=\exp\frac{1}{2}\{(U+i\tilde{U})-(V+i\tilde{V})\}\ \text{a.e.}$$

and put $\phi_2=\phi/\phi_1$ then ϕ_1 and $\bar{\phi}_2$ are invertible function in H^p for some p, $p>1$ such that $|\phi_1|^2=\exp(U-V)$ a.e. and $|\phi_2|^2=\exp(U+V)$ a.e.. Define the operator R by

$$Rf=(\phi_1^{-1}P_+ + \phi_2 P_-)(\phi_2^{-1}f),\ f \text{ is in } A+\bar{A}_0.$$

Since $\phi_2^{-1}f$ is in $L^2(We^V)\cap L^p(m)$ for some constant p, $p>1$ we have

$$\begin{aligned}\|Rf\|_W &\leqq \|\phi_1^{-1}P_+(\phi_2^{-1}f)\|_W + \|\phi_2 P_-(\phi_2^{-1}f)\|_W \\ &\leqq (\exp\|U\|_\infty)^{1/2}\{\|P_+(\phi_2^{-1}f)\|_{We^V} + \|P_-(\phi_2^{-1}f)\|_{We^V}\} \\ &\leqq 2(\exp\|U\|_\infty)^{1/2}\|P_+\|_{We^V}\|\phi_2^{-1}f\|_{We^V} \\ &\leqq 2(\exp\|U\|_\infty)\|P_+\|_{We^V}\|f\|_W.\end{aligned}$$

The third inequality holds since We^V is in (A_2). Thus R extends to a bounded map of $L^2(W)$ to $L^2(W)$. We shall show that for a function f in $A+\bar{A}_0$, $R(\phi P_+ + P_-)=(\phi P_+ + P_-)R=f$. Since $P_+\phi_1 P_+=\phi_1 P_+$, $P_-\phi_2^{-1}P_-=\phi_2^{-1}P_-$ and $P_-\phi_1 P_+=P_+\phi_2^{-1}P_-=0$, we have

$$R(\phi P_+ + P_-)f=(\phi_1^{-1}P_+ + \phi_2 P_-)(\phi_2^{-1}(\phi P_+ + P_-)f)=(\phi_1^{-1}P_+ + \phi_2 P_-)((\phi_1 P_+ + \phi_2^{-1}P_-)f)=f.$$

Since $P_+\phi_1^{-1}P_+=\phi_1^{-1}P_+$, $P_-\phi_2 P_-=\phi_2 P_-$, $P_-\phi_1^{-1}P_+=P_+\phi_2 P_-=0$, we have

$$(\phi P_+ + P_-)Rf=(\phi P_+ + P_-)(\phi_1^{-1}P_+ + \phi_2 P_-)(\phi_2^{-1}f)=f.$$

Hence $\phi P_+ + P_-$ has a bounded inverse, namely R. Hence (i) follows. The operator norm inequality follows from the proof of Proposition 6. This completes the proof.

For a W in (A_p), the necessary and sufficient conditions for T_ϕ to be invertible in $H^p(W)$ was given by Rochberg (cf.[16]). Theorem 8 is the case $p=2$. It is possible to modify this theorem for p, $1<p<\infty$.

Proposition 9. *For a weight W in (A_2), either of the following two conditions imply that $\phi P_+ + P_-$ has a dense range in $L^2(W)$.*

(a) ϕ is an outer function in H^∞.

(b) ϕ is a function in $L^\infty_R(m)$ such that (ess inf ϕ, ess sup ϕ) does not contain zero.

Proof. Since W is in (A_2), there exists an invertible function h in H^2 such that $W=|h|^2$ a.e.. Let $(\cdot,\cdot)_W$ denote the inner product in $L^2(W)$. Let g be a function in $L^2(W)$ such that $((\phi P_+ + P_-)f, g)_W=0$, for all f in $L^2(W)$. Since f_+/h is in $H^2(W)$ and $f_-/\bar{h}$ is in $\bar{H}^2_0(W)$, we have $(\phi(f_+/h), g)_W=0$ for all f_+ in A, and $((f_-/\bar{h}), g)_W=0$ for all f_- in $\bar{A}_0$. Hence $\phi\bar{h}\bar{g}$ is in H^2_0 and $\bar{h}g$ is in H^2. Put $F=Wg$ and $G=\bar{\chi}\bar{g}\phi W$, then F and G are functions in H^1 and hence FG belongs to $H^{1/2}$.

Suppose (a) holds. Since $(\chi FG)/\phi = W^2|g|^2 \geqq 0$ a.e., $(\chi FG)/\phi$ is a function in $H^{1/2}$ which is real and non-negative almost everywhere. Hence there exists a constant C such that $(\chi FG)/\phi = C$ a.e. (cf. [6], p.95). Since ϕ is an outer function, $C=0$. Since ϕ and W are non-zero functions, $g=0$ a.e.. Suppose (b) holds. Since $\chi FG = \phi W^2|g|^2$ a.e. and (ess inf ϕ, ess sup ϕ) does not contain zero, we have $\chi FG \geqq 0$ a.e. or $\chi FG \leqq 0$ a.e.. Since χFG is in $H^{1/2}$, there exists a constant C such that $\chi FG = C$ a.e.. Hence $g=0$ a.e.. This completes the proof.

Proposition 10. *Suppose ϕ is an outer function in H^∞ not equal to one. Let ε be a positive constant. For a weight W in (A_2), $\phi P_+ + P_-$ has a dense range in $L^2(W)$ and the following conditions are equivalent.*

(*i*) $\varepsilon\|f\|_W \leqq \|(\phi P_+ + P_-)f\|_W$, *for all f in $A+\bar{A}_0$.*

(*ii*) $\varepsilon \leqq \min\{1, |\phi|\}$ *a.e. and there exist a positive constant C and two real functions u, v such that*

$$W = \frac{C}{r(\varepsilon, \phi)}\exp(u - \tilde{v}) \ a.e.,$$
$$|v| \leqq \cos^{-1} r(\varepsilon, \phi) \ a.e. \ and \ |u| \leqq U(\varepsilon, \phi, v) \ a.e..$$

Proof. By Cotlar-Sadosky's theorem, it follows from (*i*) that there exists a k in H^1 such that

$$|(\phi - \varepsilon^2)W - k|^2 \leqq W^2(1-\varepsilon^2)(|\phi|^2 - \varepsilon^2) \text{ a.e..}$$

Put $g = \varepsilon^2 - \phi$, then g is in H^∞. Put $k = -\varepsilon^{-2}\phi^{-1}$, then k and k^{-1} belong to H^∞, since ϕ is an outer function and $\varepsilon \leqq |\phi|$ a.e.. Let s be any function in $A(\varepsilon^2 - \phi)$. Since Re $kg \geqq 0$ a.e. and

$$\frac{\exp(is - \tilde{s})}{|\exp(is - \tilde{s})|} = \frac{g}{|g|} \text{ a.e.,}$$

there exists a positive constant γ such that $\exp(is - \tilde{s}) = \gamma g$ a.e. (cf.[11], Proposition 5). Hence $\tilde{s} = -\log|\varepsilon^2 - \phi| + c$ a.e. for some real constant c. We use Lemma A to complete the proof.

Acknowledgements. The author wishes to thank Prof. T. Nakazi for many helpful conversations. This research was partly supported by Grant-in-Aid for Scientific Research.

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