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Algebraic Properties of Singular Integral Operators on L^2 with Cauchy Kernel

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This paper is dedicated to the memory of late Professor Takayuki Furuta

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Abstract. Let α and β be functions in $L^\infty(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. Let P denote the orthogonal projection from $L^2(\mathbb{T})$ onto the Hardy space $H^2(\mathbb{T})$, and $Q=I-P$, where I is the identity operator on $L^2(\mathbb{T})$. This paper is concerned with the singular integral operators $S_{\alpha,\beta}$ on $L^2(\mathbb{T})$ of the form $S_{\alpha,\beta}f=\alpha Pf+\beta Qf$, for $f\in L^2(\mathbb{T})$. In this paper, we study the hyponormality of $S_{\alpha,\beta}$ which is related to the Toeplitz operator on $H^2(\mathbb{T})$.

1. Introduction

For $1\leq p\leq\infty$, $L^p=L^p(\mathbb{T})$ denotes the usual Lebesgue space on the unit circle $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$ and $H^p=H^p(\mathbb{T})$ denotes the usual Hardy space on $\mathbb{T}$. If $p=2$, then $\langle f,g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{ix})\bar{g}(e^{ix})dx$ and $\|f\|=\|f\|_2$. Let $z=e^{ix}$, let $H_0^2=zH^2$, and let $H^{2\perp}=L^2\ominus H^2$. Then $H^{2\perp}=\overline{H_0^2}$. Let P denote the orthogonal projection of L^2 onto H^2. Let I denote the identity operator on L^2, and let $Q=I-P$. Then Q is an orthogonal projection of L^2 onto $H^{2\perp}$. In L^2, the sequence e_n, defined as $e_n(e^{ix})=e^{inx}$, $n\in\mathbb{Z}$, is an orthonormal sequence. Here the n-th Fourier coefficient of f is defined by $\langle f,e_n\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{ix})e^{-inx}dx=\hat{f}(n)=f_n$. Let P_0 denote the rank one orthogonal projection of L^2 onto $\mathbb{C}$ such that $(P_0f)(z)=\hat{f}(0)(f\in L^2)$. Let $I_0=I-P_0$. For $\alpha\in L^\infty$, let M_α denote the

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multiplication operator on L^2 such that $M_\alpha f=\alpha f$, $(f\in L^2)$, let T_α denote the Toeplitz operator on H^2 such that

$$T_\alpha f=P(\alpha f),\ (f\in H^2),$$

let $\tilde{T}_\alpha$ denote the operator on $H^{2\perp}$ such that

$$\tilde{T}_\alpha f=Q(\alpha f),\ (f\in H^{2\perp}),$$

let H_α denote the Hankel operator of H^2 to $H^{2\perp}$ such that

$$H_\alpha f=Q(\alpha f),\ (f\in H^2)$$

and let $\tilde{H}_\alpha$ denote the operator on $H^{2\perp}$ to H^2 such that

$$\tilde{H}_\alpha f=P(\alpha f),\ (f\in H^{2\perp}).$$

Then $\tilde{H}_\phi=H^*_{\bar{\phi}}$. For $\alpha,\beta\in L^\infty$, let $S_{\alpha,\beta}$ denote the singular integral operator on L^2 such that

$$S_{\alpha,\beta}f=\alpha Pf+\beta Qf,\ (f\in L^2).$$

Then

$$(S_{\alpha,\beta}f)(z)=\frac{\alpha(z)+\beta(z)}{2}f(z)+\frac{\alpha(z)-\beta(z)}{2}\frac{1}{\pi i}\int_{\mathbb{T}}\frac{f(\tau)}{\tau-z}d\tau,$$

where the integral is understood in the sense of Cauchy's principal value (cf. [6], p.12). If $f\in L^1$, then $(S_{\alpha,\beta}f)(z)$ exists for almost all $z\in\mathbb{T}$. The normality of $S_{\alpha,\beta}$ was established by Nakazi and the author [27]. An operator A is called hyponormal if its self-commutator $[A^*, A]=A^*A-AA^*$ is positive. When $\alpha-\beta$ is a constant, then $S_{\alpha,\beta}$ is hyponormal if and only if $S_{\alpha,\beta}$ is normal ([13]). In this paper, we study the hyponormal operator $S_{\alpha,\beta}$.

2. HYPONORMAL SI-OPERATOR

In this section, when β is a complex number, the conditions of symbols α and β of hyponormal operators $S_{\alpha,\beta}$ are determined using Toeplitz operators and Hankel operators.

Lemma 1.1. *Let α and β be in L^∞. Suppose $S_{\alpha,\beta}$ is a hyponormal operator.*
(1) *If $\bar{\alpha}$ is in H^∞, then $\bar{\beta}$ is in H^∞, and for all $f_2\in H^{2\perp}$, $\|\tilde{H}_{\bar{\alpha}}f_2\|\le\|\tilde{H}_{\bar{\beta}}f_2\|$.*
(2) *If β is in H^∞, then α is in H^∞, and for all $f_1\in H^2$, $\|H_{\bar{\beta}}f_1\|\le\|H_{\bar{\alpha}}f_1\|$.*
Proof. For all f in L^2, $S^*_{\alpha,\beta}f=P(\bar{\alpha}f)+Q(\bar{\beta}f)$. Since $S_{\alpha,\beta}$ is hyponormal, it follows that for all $f_1\in H^2$ and $f_2\in H^{2\perp}$,

$$\begin{aligned}0&\le\langle(S_{\alpha,\beta}^*S_{\alpha,\beta}-S_{\alpha,\beta}S_{\alpha,\beta}^*)(f_1+f_2),\ f_1+f_2\rangle\\&=\|S_{\alpha,\beta}(f_1+f_2)\|^2-\|S_{\alpha,\beta}^*(f_1+f_2)\|^2\\&=\|\alpha f_1+\beta f_2\|^2-\left\|P\overline{\alpha}(f_1+f_2)\right\|^2-\left\|Q\overline{\beta}(f_1+f_2)\right\|^2.\end{aligned}$$

Therefore, for all $f_1\in H^2$,

$$\begin{aligned}0&\le\left\|\overline{\alpha}f_1\right\|^2-\left\|P\overline{\alpha}f_1\right\|^2-\left\|Q\overline{\beta}f_1\right\|^2\\&=\left\|Q\overline{\alpha}f_1\right\|^2-\left\|Q\overline{\beta}f_1\right\|^2,\end{aligned}$$

and for all $f_2\in H^{2\perp}$,

$$\begin{aligned}0&\le\left\|\overline{\beta}f_2\right\|^2-\left\|P\overline{\alpha}f_2\right\|^2-\left\|Q\overline{\beta}f_2\right\|^2\\&=\left\|P\overline{\beta}f_2\right\|^2-\left\|P\overline{\alpha}f_2\right\|^2.\end{aligned}$$

Suppose $\overline{\alpha}$ is in H^∞. Since for all $f_1\in H^2$, $\left\|Q\overline{\beta}f_1\right\|\le\left\|Q\overline{\alpha}f_1\right\|$, this implies that $Q\overline{\beta}f_1=0$, and hence $\overline{\beta}$ is in H^∞. Hence (1) holds.

Suppose β is in H^∞. Since for all $f_2\in H^{2\perp}$, $\left\|P\overline{\alpha}f_2\right\|\le\left\|P\overline{\beta}f_2\right\|$, this implies that $P\overline{\alpha}f_2=0$, and hence α is in H^∞. Hence (2) holds. □

Lemma 1.2. *Let α be in L^∞, and let β be a complex number. Then for all $f_1\in H^2$ and $f_2\in H^{2\perp}$,*

$$(S_{\alpha,\beta}^*S_{\alpha,\beta}-S_{\alpha,\beta}S_{\alpha,\beta}^*)(f_1+f_2)=P|\alpha|^2f_1-\alpha P\overline{\alpha}f_1+(\beta-\alpha)P\overline{\alpha}f_2+\overline{\beta}Q\alpha f_1.$$

Proof. Let $A=S_{\alpha,\beta}$. Then

$$\begin{aligned}A^*A(f_1+f_2)&=A^*(\alpha f_1+\beta f_2)=A^*(\alpha f_1)+A^*(\beta f_2)\\&=P\overline{\alpha}\alpha f_1+Q\overline{\beta}\alpha f_1+P\overline{\alpha}\beta f_2+Q\overline{\beta}\beta f_2\\&=P|\alpha|^2f_1+\overline{\beta}Q\alpha f_1+\beta P\overline{\alpha}f_2+|\beta|^2f_2,\end{aligned}$$

and

$$\begin{aligned}AA^*(f_1+f_2)&=AP\overline{\alpha}(f_1+f_2)+AQ\overline{\beta}(f_1+f_2)\\&=\alpha P\overline{\alpha}(f_1+f_2)+\beta Q\overline{\beta}(f_1+f_2)\\&=\alpha P\overline{\alpha}f_1+\alpha P\overline{\alpha}f_2+|\beta|^2f_2.\end{aligned}$$

Hence

$$(A^*A-AA^*)(f_1+f_2)=P|\alpha|^2f_1-\alpha P\overline{\alpha}f_1+(\beta-\alpha)P\overline{\alpha}f_2+\overline{\beta}Q\alpha f_1.$$

□

Theorem 1.1. *Let α be in L^∞ and let β be a complex number. Then $S_{\alpha,\beta}$ is hyponormal if and only if T_α is analytic.*

Proof. Suppose $S_{\alpha,\beta}$ is hyponormal. Since β is a complex number, it follows from Lemma 1.1 (2), that α is in H^∞, and hence T_α is analytic. Conversely suppose T_α is analytic. Then α is in H^∞. Let $A=S_{\alpha,\beta}$. By Lemma 1.2, for all $f_1 \in H^2$ and $f_2 \in H^{2\perp}$,

$$(A^*A-AA^*)(f_1+f_2)=P|\alpha|^2f_1-\alpha P\bar{\alpha}f_1.$$

Hence

$$\begin{aligned}\langle (A^*A-AA^*)(f_1+f_2), f_1+f_2\rangle &= \langle P|\alpha|^2 f_1, f_1+f_2\rangle - \langle \alpha P\bar{\alpha} f_1, f_1+f_2\rangle \\ &= \langle P|\alpha|^2 f_1, f_1\rangle - \langle \alpha P\bar{\alpha} f_1, f_1\rangle \\ &= \|\bar{\alpha} f_1\|^2 - \|P\bar{\alpha} f_1\|^2 = \|Q\bar{\alpha} f_1\|^2 \geq 0.\end{aligned}$$

Therefore $S_{\alpha,\beta}$ is hyponormal. □

Corollary 1.1. *Let φ be in L^∞. Then $S_{\varphi,0} = M_\varphi P$ is hyponormal if and only if $S_{\varphi,1} = \varphi P + Q$ is hyponormal if and only if T_φ is analytic.*

Suppose α is a constant multiple of a unimodular function in L^∞ and β is a complex number. Then we study the conditions of symbols α and β of subnormal and quasinormal $S_{\alpha,\beta}$.

Lemma 1.3. ([13]) *For a bounded analytic function φ, the Toeplitz operator T_φ is quasinormal if and only if φ is a constant multiple of an inner function.*

Theorem 1.2. *Let α be a constant multiple of a unimodular function in L^∞ and let β be a complex number. Then $S_{\alpha,\beta}$ is subnormal if and only if $S_{\alpha,\beta}$ is hyponormal if and only if $S_{\alpha,\beta}$ is quasinormal if and only if T_α is analytic and quasinormal if and only if α is a constant multiple of an inner function.*

Proof. Let $A = S_{\alpha,\beta}$. Suppose A is subnormal. Since every subnormal operator is hyponormal, it follows that A is hyponormal. By Lemma 1.1(2), this implies that α is in H^∞. Since $|\alpha|$ is a constant, it follows that α is a constant multiple of an inner function. By Lemma 1.3, T_α is quasinormal. Conversely suppose T_α is analytic and quasinormal. By Lemma 1.3, this implies that α is a constant multiple of an inner function. By the proof of Lemma 1.2, for all $f_1 \in H^2$ and $f_2 \in H^2$,

$$\begin{aligned}A^*A(f_1+f_2)&=P|\alpha|^2f_1+\overline{\beta}Q\alpha f_1+\beta P\overline{\alpha}f_2+|\beta|^2f_2\\&=|\alpha|^2f_1+|\beta|^2f_2.\end{aligned}$$

Since α is a constant multiple of an inner function, it follows that

$$\begin{aligned}(A(A^*A)-(A^*A)A)(f_1+f_2)&=A(A^*A)(f_1+f_2)-(A^*A)(\alpha f_1+\beta f_2)\\&=A\left(|\alpha|^2f_1+|\beta|^2f_2\right)-\left(|\alpha|^2\alpha f_1+|\beta|^2\beta f_2\right)\\&=|\alpha|^2\alpha f_1+|\beta|^2\beta f_2-\left(|\alpha|^2\alpha f_1+|\beta|^2\beta f_2\right)=0.\end{aligned}$$

Hence A is quasinormal. We recall that every quasinormal operator is subnormal. Hence A is subnormal. By Theorem 1.1, this completes the proof. □

Suppose φ is a constant multiple of a unimodular function in L^∞. Then we study the conditions of symbols φ of 2-contractive (i.e. convex, c.f. [1], [3]) operators $S_{\varphi,0}=M_\varphi P$.

Lemma 1.4. *Let α and β* be *in L^∞. Suppose $S_{\alpha,\beta}$ is 2-contractive (i.e. convex).*
(1) *If $|\alpha|\geq 1$ a.e., then for all f_1 in H^2, $\|(\alpha T_\alpha+\beta H_\alpha)f_1\|\geq\|f_1\|$.*
(2) *If $|\beta|\geq 1$ a.e., then for all f_2 in $H^{2\perp}$, $\left\|\left(\alpha\tilde{H}_\beta+\beta\tilde{T}_\beta\right)f_2\right\|\geq\|f_2\|$.*
(3) *If α is a bounded analytic function, then for all f_1 in H^2, $\|f_1\|^2-2\|\alpha f_1\|^2+\|\alpha^2f_1\|^2\geq 0$.*
(4) *If $\overline{\beta}$ is a bounded analytic function, then for all f_2 in $H^{2\perp}$, $\|f_2\|^2-2\|\beta f_2\|^2+\|\beta^2f_2\|^2\geq 0$.*

Proof. (1): Let $A=S_{\alpha,\beta}$. Then A is 2-contractive (i.e. convex). For all f_1 in H^2 and f_2 in $H^{2\perp}$,

$$\|f_1+f_2\|^2-2\|A(f_1+f_2)\|^2+\|A^2(f_1+f_2)\|^2\geq 0.$$

Hence

$$\|f_1\|^2-2\|Af_1\|^2+\|A^2f_1\|^2\geq 0.$$

Since $A(f_1+f_2)=\alpha f_1+\beta f_2$ and

$$A^2(f_1+f_2)=A(\alpha f_1+\beta f_2)=\alpha P\alpha f_1+\alpha P\beta f_2+\beta Q\alpha f_1+\beta Q\beta f_2,$$

it follows that

$$\begin{aligned}0&\le\|f_1\|^2-2\|Af_1\|^2+\|A^2f_1\|^2\\&=\|f_1\|^2-2\|\alpha f_1\|^2+\|\alpha P\alpha f_1+\beta Q\alpha f_1\|^2\\&\le\|\alpha P\alpha f_1+\beta Q\alpha f_1\|^2-\|f_1\|^2\\&=\|(\alpha T_\alpha+\beta H_\alpha)f_1\|^2-\|f_1\|^2.\end{aligned}$$

(2): Since A is 2-contractive (i.e. convex), it follows that for all f_2 in $H^{2\perp}$,

$$\|f_2\|^2-2\|Af_2\|^2+\|A^2f_2\|^2\ge 0.$$

Hence

$$\begin{aligned}0&\le\|f_2\|^2-2\|Af_2\|^2+\|A^2f_2\|^2\\&=\|f_2\|^2-2\|\beta f_2\|^2+\|\alpha P\beta f_2+\beta Q\beta f_2\|^2\\&\le\|\alpha P\beta f_2+\beta Q\beta f_2\|^2-\|f_2\|^2\\&=\left\|\left(\alpha\tilde{H}_\beta+\beta\tilde{T}_\beta\right)f_2\right\|^2-\|f_2\|^2.\end{aligned}$$

(3): Since A is 2-contractive (i.e. convex), it follows that for all f_1 in H^2,

$$\begin{aligned}0&\le\|f_1\|^2-2\|Af_1\|^2+\|A^2f_1\|^2\\&=\|f_1\|^2-2\|\alpha f_1\|^2+\|\alpha^2 f_1\|.\end{aligned}$$

(4): Since A is 2-contractive (i.e. convex), it follows that for all f_2 in $H^{2\perp}$,

$$\begin{aligned}0&\le\|f_2\|^2-2\|Af_2\|^2+\|A^2f_2\|^2\\&=\|f_2\|^2-2\|\beta f_2\|^2+\|\beta^2 f_2\|^2.\end{aligned}$$

□

Theorem 1.3. *Let* φ be a constant multiple of a unimodular function in L^∞. *Suppose an operator* $S_{\varphi,0}=M_\varphi P$ *is 2-contractive (i.e. convex, c.f. [1], [3]). Then* $|\varphi|\ge 1$ *a.e. and* $|\varphi|\cdot\|T_\varphi f_1\|\ge\|f_1\|$ *for all* f_1 *in* H^2.

Proof. Let $A=S_{\varphi,0}$. Since A is 2-contractive (i.e. convex), it follows from Lemma 1.4(1), for all f_1 in H^2, $\frac{1}{|\varphi|}\|f_1\|\le\|T_\varphi f_1\|\le|\varphi|\cdot\|f_1\|$. Hence $|\varphi|\ge 1$ a.e. □

Definition 1.1. *For* $0<p<\infty$, *A belongs to class B(p) if* $(A^*A)^p=A^{*p}A^p$.

By the elementary calculation in the proof of the following corollary, it follows that if A is contractive and belongs to class $B(2)$, then A is 2-contractive.

Corollary 1.2. *Let φ be a unimodular function in L^∞. Suppose $S_{\varphi,0}=M_\varphi P$ is quasinormal. Then $S_{\varphi,0}$ is 2-contractive (i.e. convex), $S_{\varphi,0}$ is contractive and belongs to class $B(2)$.*

Proof. Suppose $A = S_{\varphi,0}$ is quasinormal. By Theorem 1.2, φ is an inner function. For all f in L^2, $\|Af\|=\|\varphi Pf\|=\|Pf\|\leq\|f\|$. Therefore A is contractive. Since $A(A^*A) = (A^*A)A$, it follows that $(A^*A)^2 = A^{*2}A^2$, and hence A is contractive and belongs to class $B(2)$. Suppose A is contractive and belongs to class $B(2)$. Then $I - A^*A$ is a positive operator. Hence, for all f in L^2,

$$\begin{aligned}\langle (I-2A^*A+A^{*2}A^2)f, f\rangle &= \langle (I-2A^*A+(A^*A)^2)f, f\rangle \\ &= \langle (I-A^*A)^2 f, f\rangle \\ &= \langle (I-A^*A)f, (I-A^*A)f\rangle \\ &= \|(I-A^*A)f\|^2 \geq 0.\end{aligned}$$

Therefore A is 2-contractive (i.e. convex). □

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