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重み付きルベーグ・ヒルベルト空間上の 解析射影の有界性について

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Boundedness of Analytic Projections on Weighted Lebesgue-Hilbert Spaces

Takanori YAMAMOTO

This paper is dedicated to the memory of late Professor Takahiko Nakazi

Abstract. An important case of non-positive operators, for which the classical theorems still hold is exhibited by the theory of Hilbert transforms. The (ordinary) Hilbert transform Hf of the function $f(x)$, $(-\infty < x < \infty)$, is defined by

$$(I) \quad Hf(x) = \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt.$$

Hf is understood as the limit, as $\varepsilon \rightarrow 0$, of $H_\varepsilon f$, where $H_\varepsilon f$ is defined for each $\varepsilon > 0$ by

$$(I a) \quad H_\varepsilon f(x) = \int_{|t-x|>\varepsilon} \frac{f(t)}{x-t} dt = \int_{-\infty}^{x-\varepsilon} \frac{f(t)}{x-t} dt + \int_{x+\varepsilon}^{\infty} \frac{f(t)}{x-t} dt.$$

Lusin, Privaloff and Plessner proved the pointwise convergence of $H_\varepsilon f$; for every $f \in L^p$, $p \geq 1$, the limit

$$(II) \quad \lim_{\varepsilon \rightarrow 0} H_\varepsilon f(x) = Hf(x)$$

exists for almost all x . The limit function $Hf(x)$ is then taken as the definition of the singular integral (I).

While the function $Hf(x)$ exists, it may not be integrable. M. Riesz has shown that if $f \in L^p$, and

$p > 1$ then also $Hf \in L^p$ and $H_\varepsilon f$ converges to Hf in the p^{th} -mean, i.e.

$$(III) \quad \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} |Hf(x) - H_\varepsilon f(x)|^p dx = 0, \quad \text{for } f \in L^p, p > 1.$$

Moreover, the following inequality of M. Riesz

$$(IV) \quad \int_{-\infty}^{\infty} |Hf(x)|^p dx \leq O_p \int_{-\infty}^{\infty} |f(x)|^p dx, \quad (p > 1),$$

holds for any $f \in L^p$, where O_p depends on p alone. (c.f. Cotlar [21])

This paper is concerned with the boundedness of the Hilbert transform and the analytic projection between two weighted Lebesgue-Hilbert spaces.

§ 1. $p=2$ のとき, 単位円周上の Koosis の定理

P. Koosis は, 次の定理 A と定理 B を示した。

[定理 A] ([62], [65])

$W \geq 0$, $W \in L^1(dx)$, $\alpha > 0$, dx は実軸 $\mathbb{R}$ 上の Lebesgue 測度

次の (i) ~ (iv) は同値である。

$$(i) \quad \exists U \geq 0 \quad \text{a.e.} \quad \int_{-\infty}^{\infty} U dx > 0 \quad \text{s.t.}$$

$$\int_{-\infty}^{\infty} |Hf|^2 U dx \leq \int_{-\infty}^{\infty} |f|^2 W dx, \quad \forall f(x) = \sum_{|\lambda| \geq \alpha} C_\lambda e^{i\lambda x}$$

$$(ii) \quad \exists \Psi(z) : \text{指数型高々 } \alpha \text{ の整関数} \quad \text{s.t.}$$

$$\int_{-\infty}^{\infty} \frac{|\Psi|^2}{W} \frac{dx}{1+x^2} < \infty$$

$$(iii) \quad \exists \varphi \in H^1 \text{ outer}, \quad |\varphi| = W \quad \text{a.e.} \quad \exists g \neq 0, \quad g \in H^\infty \quad \text{s.t.}$$

$$\left\| e^{2iax} \frac{|\varphi|}{\varphi} - g \right\|_\infty \leq 1$$

$$(iv) \quad \exists \varphi \in H^1 \text{ outer}, \quad |\varphi| = W \quad \text{a.e.} \quad \text{s.t.}$$

$$e^{2iax} \frac{|\varphi|}{\varphi} + H^\infty \text{ は, } L^\infty/H^\infty \text{ の単位球の端点でない。}$$

[定理 B] ([63])

$W \geq 0$, $W \in L^1(d\theta)$, $d\theta$ は単位円周 $\mathbb{T}$ 上の Lebesgue 測度

次の (i), (ii) は同値である。

$$(i) \exists U \geq 0 \text{ a.e. } \int_{\mathbb{T}} U d\theta > 0 \text{ s.t.}$$

$$\int_{\mathbb{T}} |Hf|^2 U d\theta \leq \int_{\mathbb{T}} |f|^2 W dx, \quad \forall f = \sum_{|n| \geq 0} c_n e^{in\theta} \text{ trigonometric polynomial}$$

$$(ii) \int_{\mathbb{T}} \frac{1}{W} d\theta < \infty$$

[定義]

$\mathbb{T}$: 単位円周, $\mathbb{Z}$: 整数全体

$d\theta$: $\mathbb{T}$ 上の正規化された Lebesgue 測度

$W(\theta) \geq 0, \in L^1(d\theta)$

$\mathcal{P}$: 正則多項式全体 i.e. $\mathcal{P} = \text{span}\{e^{in\theta}; n \geq 0, \in \mathbb{Z}\}$

$a, b \in \mathbb{Z}: a \leq -1 < 0 \leq b$ を充たす。

T : 三角多項式全体の上で定義された作用素

$$(T, W, (a, b)) \equiv \left\{ U \left| \begin{array}{l} U(\theta) \geq 0 \text{ a.e. } \mathbb{T} \\ \int_{\mathbb{T}} |Tf|^2 U d\theta \leq \int_{\mathbb{T}} |f|^2 W d\theta \\ \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}} \end{array} \right. \right\}$$

一般に, $f \in \mathcal{D} \subseteq \{\text{三角多項式全体}\}$ なる時は, $(T, W, \mathcal{D})$ と書くことがある。

作用素 T として, 次のような作用素を考える。

“ $\widehat{}$ ”は Fourier 変換を意味する。

$$H: \text{Hilbert 変換} \quad \widehat{Hf}(k) = \begin{cases} -i \widehat{f}(k) & k \geq 0, k \in \mathbb{Z} \\ i \widehat{f}(k) & k \leq -1 \end{cases}$$

$$P: \text{正則射影} \quad \widehat{Pf}(k) = \begin{cases} \widehat{f}(k) & k \geq 0 \\ 0 & k \leq -1 \end{cases}$$

$$P_E: E \subset \mathbb{Z}: \text{有限集合} \quad \widehat{P_E f}(k) = \begin{cases} \widehat{f}(k) & k \in E \\ 0 & k \notin E \end{cases}$$

特に, $P_n \equiv P_{[0, n]}$ とする。

$$Hf(\theta) = \text{p.v.} \int_{\mathbb{T}} f(t) \left(1 + \cot \frac{t - \theta}{2} \right) dt \text{ と書ける。}$$

但し, $\text{p.v.} \int$ は Cauchy の主値積分を表わす。

[定理 1]

$W \geq 0, \in L^1(d\theta), a, b \in \mathbb{Z}, a \leq -1, 0 \leq b$

次の (i) ~ (vii) は同値である。

$$(i) (H, W, (a, b)) \neq \{0\}$$

$$(i)' (H, W, (a, b)) \ni \exists U \text{ s.t. } \log U \in L^1(d\theta)$$

$$(ii) (P, W, (a, b)) \neq \{0\}$$

$$(ii)' (P, W, (a, b)) \ni \exists U \text{ s.t. } \log U \in L^1(d\theta)$$

$$(iii) \forall n \geq 0, (P_n, W, (a, b)) \ni \exists U_n \neq 0 \text{ a.e.}$$

$$(iii)' \forall n \geq 0, \exists K_n : \text{定数 s.t.}$$

$$\int_{\mathbb{T}} |P_n f|^2 W d\theta \leq K_n \int_{\mathbb{T}} |f|^2 W d\theta, \quad \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}$$

$$(iv) \forall E \subset \mathcal{P} : \text{有限次元部分空間 s.t. } e^{ib\theta} \in E \text{ について,}$$

$$(P_E, W, (a, b)) \ni \exists U_E \neq 0 \text{ a.e.}$$

$$(iv)' \forall E \subset \mathcal{P} : \text{有限次元部分空間 s.t. } e^{ib\theta} \in E \text{ について,}$$

$$\sup_{e^{ik\theta} \in E} \left| \widehat{f}(k) \right| \leq K_E \int_{\mathbb{T}} |f|^2 W d\theta, \quad \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}$$

$$(v) \inf_{f \in e^{i(b+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}} \int_{\mathbb{T}} |e^{ib\theta} - f|^2 W d\theta > 0$$

$$(vi) \exists \Psi(e^{i\theta}) \in \mathcal{P} : b-a-1 \text{ 次 s.t. } \int_{\mathbb{T}} \frac{|\Psi|^2}{W} d\theta < \infty$$

$$(vii) \exists \varphi \in H^1 \text{ outer, } |\varphi(\theta)| = W(\theta) \text{ a.e. } \exists g \neq 0, \in H^\infty$$

$$\text{s.t. } \left\| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g \right\|_\infty \leq 1$$

$$(i) \leftrightarrow (ii), (i)' \leftrightarrow (ii)' \text{ の証明 :}$$

$2P = I + iH$ より明らか。特に $(H, W, (a, b)) \subseteq (P, W, (a, b))$ に注意する。

$$(ii)' \rightarrow (iii) \text{ の証明 :}$$

$$(ii)' \text{ より, } \exists U, \log U \in L^1(d\theta) \text{ s.t. } \int |Pf|^2 U d\theta \leq \int |f|^2 W d\theta$$

$$\text{従って, } \int_{\mathbb{T}} |P_n f|^2 U d\theta \leq C \int_{\mathbb{T}} |Pf|^2 U d\theta, \quad \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}$$

を示せばよい。すなわち,

$$\mathcal{M} \equiv \text{span}\{e^{ik\theta} ; b \leq k \leq n\}, \quad \mathcal{N} \equiv \text{span}\{e^{ik\theta} ; n+1 \leq k\}$$

$$\text{について, } \|x\|_{L^2(U)} \leq C \|x+y\|_{L^2(U)}, \quad \forall x \in \mathcal{M}, \quad \forall y \in \mathcal{N}$$

を示せばよい。

$$\text{最初に, } [\mathcal{M}]_{L^2(U)} \cap \mathcal{M} = \{0\} \text{ を示す。}$$

$$\odot U \geq 0, \log U \in L^1(d\theta) \text{ より, } \exists h \in H^2 \text{ outer s.t. } |h|^2 = U \text{ a.e.}$$

$$\text{Beurling の定理より, } [h\mathcal{P}]_{L^2(d\theta)} = H^2$$

$$\text{一方, 明らかに, } [h\mathcal{P}]_{L^2(d\theta)} = h[\mathcal{P}]_{L^2(U)}$$

$$\therefore [\mathcal{P}]_{L^2(U)} = h^{-1} H^2 \subseteq N_+$$

$$L^\infty(d\theta) \cap N_+ = H^\infty \text{ より, } L^\infty(d\theta) \cap [\mathcal{P}]_{L^2(U)} \subseteq H^\infty$$

もし, $w \in [\mathcal{N}]_{L^2(U)} \cap \mathcal{M}$ ならば,

$$\exists y_j \in \mathcal{N} \quad \text{s.t.} \quad y_j \rightarrow w \text{ in } L^2(U)$$

$$e^{-i(n+1)\theta} y_j \in \mathcal{P} \text{ より, } e^{-i(n+1)\theta} w \in [\mathcal{P}]_{L^2(U)} \cap L^\infty(d\theta) \subseteq H^\infty$$

$$\therefore w \in \mathcal{M} \cap e^{i(n+1)\theta} H^\infty = \{0\}$$

$$\therefore w = 0$$

さて, 上の不等式が不成立と仮定する。

$$\exists \{x_j\} \subseteq \mathcal{M}, \quad \exists \{y_j\} \subseteq \mathcal{N} \quad \text{s.t.}$$

$$\|x_j\| = 1 \text{ 且 } x_j + y_j \rightarrow 0 \text{ in } L^2(U)$$

$\{x_j\}$ は有限次元空間 $\mathcal{M}$ の有界列ゆえ, 収束部分列を持つ。

それを改めて $\{x_j\}$ と書く。

$$\therefore x_j \rightarrow x \in \mathcal{M} \text{ in } L^2(U) \quad \therefore y_j \rightarrow -x \text{ in } L^2(U)$$

$$y_j \in \mathcal{N} \text{ より, } x \in [\mathcal{N}]_{L^2(U)} \cap \mathcal{M} = \{0\} \quad \therefore x = 0$$

一方, $\|x\| = \lim_j \|x_j\| = 1$ 矛盾。

(iii) $\rightarrow$ (iii)' の証明:

有限次元空間では, 全てのノルムは同値ゆえ, 特に $L^2(U)$ ノルムと, $L^2(W)$ ノルムも同値である。

従って,

$$\int_{\mathbb{T}} |P_n f|^2 W d\theta \leq K_n \int_{\mathbb{T}} |P_n f|^2 U d\theta, \quad f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}$$

$$(iii) \text{ より, } \leq K_n \int_{\mathbb{T}} |f|^2 W d\theta$$

(iii) $\rightarrow$ (iv) の証明:

$$\forall E \subset \mathcal{P}: \text{有限次元部分空間} \quad \text{s.t.} \quad E \ni e^{ib\theta}$$

$$\text{について, } \exists n \geq 0 \quad \text{s.t.} \quad E \subseteq \text{span}\{e^{ik\theta}; 0 \leq k \leq n\}$$

$$\text{ゆえに, } \int_{\mathbb{T}} |P_E f|^2 U d\theta \leq C \int_{\mathbb{T}} |P_n f|^2 U d\theta, \quad \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}$$

を示せばよい。すなわち,

$$\mathcal{M} \equiv \text{span}\{e^{ib\theta}; b \leq k \leq n, \text{ 且 } k \in E\}$$

$$\mathcal{N} \equiv \text{span}\{e^{ib\theta}; b \leq k \leq n, \text{ 且 } k \notin E\}$$

について,

$$\|x\|_{L^2(U)} \leq C \|x + y\|_{L^2(U)}, \quad \forall x \in \mathcal{M}, \quad \forall y \in \mathcal{N}$$

$\mathcal{N}$ も有限次元より, $\mathcal{N} = [\mathcal{N}]_{L^2(U)}$ となり, あとは, (ii)' $\rightarrow$ (iii) の証明と同様。

(iv) $\rightarrow$ (iv)' の証明:

有限次元空間では, 全てのノルムは同値ゆえ,

$$\sup_{e^{ik\theta} \in E} |\widehat{f}(k)| = \sup_{e^{ik\theta} \in E} |\widehat{P_E f}(k)| \leq \exists K_E \int_{\mathbb{T}} |P_E f|^2 U \, d\theta$$

$$(iv) \text{ より, } \leq K_E \int_{\mathbb{T}} |f|^2 W \, d\theta$$

(iv)' $\rightarrow$ (v) の証明:

$$e^{ib\theta} \in E \text{ より, } |\widehat{f}(b)| \leq \sup_{e^{ik\theta} \in E} |\widehat{f}(k)| \stackrel{(iv)'}{\leq} \exists K_E \int_{\mathbb{T}} |f|^2 W \, d\theta$$

$$\therefore \inf_{f \in e^{i(b+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}} \int_{\mathbb{T}} |e^{ib\theta} - f|^2 W \, d\theta \geq \frac{1}{K_E} > 0$$

(v) $\rightarrow$ (vi) の証明:

$$(v) \text{ より, } e^{ib\theta} \notin [e^{i(b+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}]_{L^2(W)} \text{ である,}$$

Hahn-Banach の分離定理より,

$$\exists g \neq 0 \in L^2(W) \text{ s.t. } \int_{\mathbb{T}} e^{-in\theta} g W \, d\theta = 0, \quad \forall n \notin (a, b]$$

一方, $W \in L^1(d\theta)$ である, Schwarz の不等式より, $gW \in L^1(d\theta)$

$$\therefore gW = \sum_{n \in (a, b]} c_n e^{in\theta}, \quad \Psi \equiv e^{-i(a+1)\theta} gW$$

$$\text{この時, } \int_{\mathbb{T}} \frac{|\Psi|^2}{W} \, d\theta = \int_{\mathbb{T}} |g|^2 W \, d\theta < \infty$$

(vi) $\rightarrow$ (vii) の証明:

Ψ は正則多項式である, $\log |\Psi| \in L^1(d\theta)$ 。一方,

$$W \geq \log W = \log \frac{W}{|\Psi|^2} + \log |\Psi|^2 \geq -\frac{|\Psi|^2}{W} + 2 \log |\Psi| \quad \text{a.e.}$$

$$W, \frac{|\Psi|^2}{W} \in L^1(d\theta) \text{ より, } \log W \in L^1(d\theta)$$

$$\therefore \exists \varphi \in H^1 \text{ outer s.t. } |\varphi| = W \quad \text{a.e.}$$

$$\frac{|\Psi|^2}{W} \in L^1(d\theta) \text{ より, } G(z) \equiv \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \frac{|\Psi|^2}{W} \, d\theta$$

$G(z)$ は, $\{|z| < 1\}$ で正則かつ, $\operatorname{Re} G(z) > 0$ である, outer である。

$$\text{この時, } \operatorname{Re} G(e^{i\theta}) = \frac{|\Psi(e^{i\theta})|^2}{W(\theta)} \quad \text{a.e.}$$

$$\text{一方, } \forall u \in H^\infty, \|u\|_\infty \leq 1 \text{ について, } \operatorname{Re} \frac{1+u(\theta)}{1-u(\theta)} \geq 0 \quad \text{a.e. である,}$$

$$\left| \frac{e^{i(b-a-1)\theta} |\Psi|^2}{\left(\frac{1+u}{1-u} + G\right)\varphi} \right| = \left| \frac{\operatorname{Re} G}{\operatorname{Re} \left(\frac{1+u}{1-u} + G\right)} \right| \leq 1 \quad \text{a.e.}$$

$$\therefore g \equiv \frac{e^{i(b-a-1)\theta} |\Psi|^2}{\left(\frac{1+u}{1-u} + G\right)\varphi} \in L^\infty(d\theta) \cap N_+ = H^\infty$$

この時,

$$\left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right| = \left| 1 - \frac{\operatorname{Re} G}{\frac{1+u}{1-u} + G} \right| = \left| \frac{\frac{1+u}{1-u} - \bar{G}}{\frac{1+u}{1-u} + G} \right| \leq 1 \quad \text{a.e.}$$

(vii) $\rightarrow$ (ii)' の証明 :

$$\begin{cases} \exists g \neq 0, \in H^\infty \quad \text{s.t.} \quad \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right| \leq 1 \quad \text{a.e.} \\ \Rightarrow \\ \exists g \neq 0, \in H^\infty \quad \text{s.t.} \quad \log \left(1 - \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right| \right) \in L^1(d\theta) \end{cases}$$

$$\therefore \rho \equiv 1 - \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right|, \quad \log \rho \in L^1(d\theta)$$

$$\therefore \left| \int_{\mathbb{T}} e^{i(b-a)\theta} \frac{|\varphi|}{\varphi} F d\theta \right| = \left| \int_{\mathbb{T}} \left(e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right) e^{i\theta} F d\theta \right|$$

$$\leq \int_{\mathbb{T}} (1-\rho) |F| d\theta, \quad \forall F \in H^1$$

$$\therefore \left| \int_{\mathbb{T}} e^{i(b-a)\theta} W G d\theta \right| \leq \int_{\mathbb{T}} (1-\rho) W |G| d\theta, \quad \forall G \in H^\infty$$

$$\therefore \left| \int_{\mathbb{T}} f \bar{g} W d\theta \right| \leq \int_{\mathbb{T}} |f \bar{g}| (1-\rho) W d\theta, \quad \begin{matrix} \forall f \in e^{ib\theta} \frac{H^\infty}{H^\infty} \\ \forall g \in e^{ia\theta} \frac{H^\infty}{H^\infty} \end{matrix}$$

$$\therefore \int_{\mathbb{T}} |f+g|^2 W d\theta = \int_{\mathbb{T}} (|f|^2 + |g|^2) W d\theta + 2 \operatorname{Re} \int_{\mathbb{T}} f \bar{g} W d\theta$$

$$\geq \int_{\mathbb{T}} (|f|^2 + |g|^2) W d\theta - 2 \int_{\mathbb{T}} |f \bar{g}| (1-\rho) W d\theta$$

$$= \int_{\mathbb{T}} ((1-\rho)|f| + |g|)^2 W d\theta + \int_{\mathbb{T}} (1-(1-\rho)^2) |f|^2 W d\theta$$

$$\geq \int_{\mathbb{T}} |f|^2 \left(1 - \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right|^2 \right) W d\theta$$

$$\therefore \left\{ U = \left(1 - \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right|^2 \right) W ; \exists g \in H^\infty \quad \text{s.t.} \quad \left\| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right\|_\infty \leq 1 \right\}$$

$$\subseteq (P, W, (a, b))$$

(ii) $\rightarrow$ (vii) の証明: $\mathcal{P} \equiv \{\text{analytic polynomial 全体}\}$

$$\int_{\mathbb{T}} |f|^2 U \, d\theta \leq \int_{\mathbb{T}} |f+g|^2 W \, d\theta, \quad \begin{array}{l} \forall f \in e^{ib\theta} \mathcal{P} \\ \forall g \in e^{ia\theta} \overline{\mathcal{P}} \end{array}$$

一般に $W \in L^1(d\theta)$ の時

$$[\mathcal{P}]_{L^2(W)} \supset H^\infty \text{ へ,}$$

$$\forall f \in e^{ib\theta} H^\infty, \quad \forall g \in e^{ia\theta} \overline{H^\infty} \text{ について成り立つ。}$$

この時, 因数分解定理より,

$$\forall F \in H^\infty \quad \exists B : \text{Blaschke product}, \quad \exists \Phi \in H^\infty : \{|z| < 1\} \text{ に zero を持たない。}$$

$$\text{s.t. } F = B \Phi^2$$

$$\therefore e^{i(b-a)\theta} F = (e^{ib\theta} \Phi) \overline{(e^{ia\theta} B \Phi)} = f \bar{g}$$

$$\left| \int_{\mathbb{T}} e^{i(b-a)\theta} F W \, d\theta \right| = \left| \int_{\mathbb{T}} f \bar{g} W \, d\theta \right| \leq \frac{1}{2} \int_{\mathbb{T}} |f|^2 (W - U) \, d\theta + \frac{1}{2} \int_{\mathbb{T}} |g|^2 W \, d\theta$$

$$= \int_{\mathbb{T}} |F| \left(W - \frac{U}{2} \right) d\theta$$

$$\therefore \left| \int_{\mathbb{T}} e^{i(b-a)\theta} \frac{|\varphi|}{\varphi} (\varphi F) d\theta \right| \leq \int_{\mathbb{T}} |\varphi F| \left(1 - \frac{U}{2W} \right) d\theta$$

$$\{\varphi F : F \in H^\infty\} \subseteq H^1 \text{ dense へ,}$$

$$\forall G \in H^1, \quad \left| \int_{\mathbb{T}} e^{i(b-a)\theta} \frac{|\varphi|}{\varphi} G \, d\theta \right| \leq \int_{\mathbb{T}} |G| \left(1 - \frac{U}{2W} \right) d\theta$$

Hahn-Banach の定理より,

$$\exists g \in H^\infty \quad \text{s.t.} \quad \operatorname{ess\,sup}_{-\pi \leq \theta < \pi} \frac{\left| e^{i(b-a)\theta} \frac{|\varphi|}{\varphi} - e^{i\theta} g \right|}{1 - \sigma} \leq 1'$$

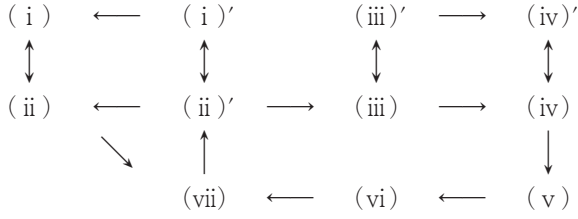
$$\text{但し, } \sigma \equiv \frac{U}{2W}$$

$$\sigma \neq 0 \text{ a.e. へ, } g \neq 0$$

$$\therefore \exists g \neq 0, \in H^\infty \text{ s.t. } \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - g \right| \leq 1 \text{ a.e.}$$

以上で定理 1 の証明が完成した。 ■

証明した順序は次の通りである。



○ (ii)' → (v) も次のように証明できる。

$$\delta \equiv \inf_{f \in e^{i(b+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}} \int_{\mathbb{T}} |e^{ib\theta} - f|^2 W d\theta$$

この時, $\delta > 0$ を示せばよい。

そこで, $\delta = 0$ と仮定すると,

$$\exists \{f_n\} \subseteq e^{i(b+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}} \quad \text{s.t.} \quad \int_{\mathbb{T}} |e^{ib\theta} - f_n|^2 W d\theta \xrightarrow{n \rightarrow \infty} 0$$

$$\text{(ii)'} \text{ より, } \int_{\mathbb{T}} |e^{ib\theta} - Pf_n|^2 U d\theta \xrightarrow{n \rightarrow \infty} 0 \text{ 且 } \log U \in L^1(d\theta)$$

なる U が存在する。

これは, Szegő の定理に矛盾する。

○ (v) → (iii) も次のように証明できる。

(v) は, $e^{ib\theta} \notin [e^{i(b+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}]_{L^2(W)}$ と同値である。

この時, $e^{i(b+1)\theta} \notin [e^{i(b+2)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}]_{L^2(W)}$ となる。

「なぜなら, もし $\in$ が成り立つならば, 明らかに,

$$e^{ib\theta} \in [e^{i(b+1)\theta} \mathcal{P} + e^{i(a-1)\theta} \overline{\mathcal{P}}]_{L^2(W)} \text{ となり, (v) に矛盾。}$$

以下, 帰納的に, $e^{ib\theta}, e^{i(b+1)\theta}, \dots, e^{in\theta} \notin [e^{i(n+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}]_{L^2(W)}$

となり, これは, (iii) に他ならない。

○ (vi) → (ii)' も次のように証明できる。

$$\frac{|\Psi|^2}{W} \in L^1(d\theta) \text{ より, } \exists G : \text{outer} \quad \text{s.t.} \quad \operatorname{Re} G = \frac{|\Psi|^2}{W} \quad \text{a.e.}$$

$$u \in H^\infty, \quad \|u\|_\infty \leq 1$$

$$\rho \equiv 1 - \left| 1 - \frac{2\operatorname{Re} G}{\frac{1+u}{1-u} + G} \right| = 1 - \left| \frac{\frac{1+u}{1-u} - \overline{G}}{\frac{1+u}{1-u} + G} \right| \geq 0 \quad \text{a.e.}$$

$\rho \equiv 0$ となるのは, u が inner の時に限る。

$e^{i(b-a-1)\theta} \overline{\Psi}$ は analytic polynomial で,

$$\left| \frac{\overline{w}^2}{\frac{1+u}{1-u}+G} \right| \leq \frac{|\overline{w}|^2}{\operatorname{Re} G} = W \quad \text{a.e. } \theta \quad \frac{e^{i(b-a-1)\theta} |\overline{w}|^2}{\frac{1+u}{1-u}+G} \in H^1$$

$\therefore \forall f \in e^{ib\theta} H^\infty$ analytic polynomial

$\forall g \in e^{ia\theta} \overline{H^\infty}$ anti-analytic polynomial

について,

$$\operatorname{Re} \int_{\mathbb{T}} f \overline{g} W \, d\theta = \operatorname{Re} \int_{\mathbb{T}} \left(1 - \frac{2 \operatorname{Re} G}{\frac{1+u}{1-u}+G} \right) f \overline{g} W \, d\theta$$

$$\leq \int_{\mathbb{T}} |f \overline{g}| (1-\rho) W \, d\theta$$

$$\therefore \int_{\mathbb{T}} |f+g|^2 W \, d\theta = \int_{\mathbb{T}} (|f|^2 + |g|^2) W \, d\theta + 2 \operatorname{Re} \int_{\mathbb{T}} f \overline{g} W \, d\theta$$

$$\geq \int_{\mathbb{T}} (|f|^2 + |g|^2) W \, d\theta - 2 \int_{\mathbb{T}} |f \overline{g}| (1-\rho) W \, d\theta$$

$$= \int_{\mathbb{T}} ((1-\rho)|f| + |g|)^2 W \, d\theta + \int_{\mathbb{T}} (1-(1-\rho)^2) |f|^2 W \, d\theta$$

$$\geq \int_{\mathbb{T}} |f|^2 (1-(1-\rho)^2) W \, d\theta$$

$$\therefore \left\{ U = \frac{4(\operatorname{Re} G) \left(\operatorname{Re} \frac{1+u}{1-u} \right)}{\left| G + \frac{1+u}{1-u} \right|^2} W \ ; \ u \in H^\infty, \ \|u\|_\infty \leq 1 \right\}$$

$$\subseteq (P, W, (a, b))$$

$$\circ \tau \equiv 1 - \left| 1 - \frac{\operatorname{Re} G}{\frac{1+u}{1-u}+G} \right| \stackrel{\rho \geq 0 \text{ より}}{\geq} 1 - \sqrt{1 - \left| \frac{\operatorname{Re} G}{\frac{1+u}{1-u}+G} \right|^2} \geq \frac{1}{2} \left| \frac{\operatorname{Re} G}{\frac{1+u}{1-u}+G} \right|^2$$

この τ についても, ρ と同様の事が言える。

$\circ u$: not inner $\Rightarrow \log(1-\rho) \in L^1(d\theta)$ を示す。

$$\odot \left| \int_{\mathbb{T}} f \overline{g} w \, d\theta \right| \leq \int_{\mathbb{T}} |f \overline{g}| (1-\rho) W \, d\theta$$

$$\therefore \left| \int_{\mathbb{T}} \frac{1}{1-\rho} f \overline{g} (1-\rho) W \, d\theta \right| \leq \left\{ \int_{\mathbb{T}} |f|^2 (1-\rho) W \, d\theta \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{T}} |g|^2 (1-\rho) W \, d\theta \right\}^{\frac{1}{2}}$$

もし, $\log(1-\rho) \notin L^1(d\theta)$ ならば, $\log W \in L^1(d\theta)$ より,

$\log(1-\rho) W \notin L^1(d\theta)$ となる。

$g=e^{ia\theta}$ とおくと,

$$\left| \int_{\mathbb{T}} \frac{1}{1-\rho} e^{-ia\theta} f(1-\rho) W d\theta \right| \leq \left\{ \int_{\mathbb{T}} |f|^2 (1-\rho) W d\theta \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{T}} (1-\rho) W d\theta \right\}^{\frac{1}{2}}$$

Szegő の定理より, $\forall f \in L^2((1-\rho)W)$ について成り立つ。

$$\therefore \int_{\mathbb{T}} \left(\frac{1}{1-\rho} \right)^2 (1-\rho) W d\theta \leq \int_{\mathbb{T}} (1-\rho) W d\theta$$

$$\therefore \int_{\mathbb{T}} \left(\frac{\rho(2-\rho)}{1-\rho} \right) W d\theta = 0 \quad \therefore \int_{\mathbb{T}} \rho W d\theta = 0$$

$$\therefore \int_{\mathbb{T}} \rho d\theta = 0 \quad \therefore \rho \equiv 0 \quad \text{a.e.}$$

$$\text{一方, } \rho = 1 - \frac{\left| 1 - \frac{1-|u|^2}{|1-u|^2} \frac{W}{|F|^2} \right|}{\left| 1 + \frac{1-|u|^2}{|1-u|^2} \frac{W}{|F|^2} \right|}$$

$$\therefore |u(e^{i\theta})| \equiv 1 \quad \text{a.e.}$$

$u \in H^\infty$ より u : inner 矛盾。

§ 2. $p=2$ のとき, 単位円周上の Helson-Sarason の定理

[命題 1] (Helson-Sarason [46])

$W \geq 0, \in L^1(d\theta), a \leq -1 < 0 \leq b$

次の (i) ~ (vi) は同値である。

$$(i) \quad \int_{\mathbb{T}} |Hf|^2 W d\theta \leq C^2 \int_{\mathbb{T}} |f|^2 W d\theta, \quad \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}$$

$$(ii) \quad \int_{\mathbb{T}} |Pf|^2 W d\theta \leq K^2 \int_{\mathbb{T}} |f|^2 W d\theta, \quad \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}$$

$$(iii) \quad \inf \int_{\mathbb{T}} |f+g|^2 W d\theta = \tau^2 > 0, \quad \text{where}$$

$$\int_{\mathbb{T}} |f|^2 W d\theta = \int_{\mathbb{T}} |g|^2 W d\theta = 1,$$

$$f \in e^{ib\theta} \mathcal{P}, \quad g \in e^{ia\theta} \overline{\mathcal{P}}$$

$$(iv) \quad \sup \left| \int_{\mathbb{T}} f \bar{g} W d\theta \right| = \rho < 1, \quad \text{where}$$

$$\int_{\mathbb{T}} |f|^2 W d\theta = \int_{\mathbb{T}} |g|^2 W d\theta = 1,$$

$$f \in e^{ib\theta} \mathcal{P}, \quad g \in e^{ia\theta} \overline{\mathcal{P}}$$

$$(v) \quad \exists \varphi \in H^1 \quad \text{outer, } |\varphi(\theta)| = W(\theta) \quad \text{a.e.}$$

$$\text{s.t.} \quad \left\| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + H^\infty \right\| = \rho < 1$$

$$(vi) \quad \exists \Psi \in : b-a-1 \text{ 次}, \quad \exists \mu, \nu \in L^\infty(d\theta)$$

$$\text{s.t.} \quad \|\nu\|_\infty = \frac{\pi}{2} - \varepsilon, \quad W(\theta) = |\Psi(\theta)|^2 e^{\mu(\theta) + \bar{\nu}(\theta)} \quad \text{a.e.}$$

$$\text{特に, } \|P\| = \inf K = \frac{1}{\sqrt{1-\rho^2}}, \quad \cos \varepsilon \leq \rho$$

$$\text{一方, } a = -b \neq 0 \text{ の時, } \|H\| = \sqrt{\frac{1+\rho}{1-\rho}} \text{ となる。}$$

証明

$$(ii) \rightarrow (iii) : \text{明らか。} \tau \geq K^{-1}$$

$$(iii) \rightarrow (ii) : \text{ここは, Forelli [32] が示した。}$$

$$0 < \tau \leq \left\| \frac{f}{\|f\|} + \frac{g}{\|g\|} \right\| \leq \left\| \frac{f}{\|f\|} + \frac{g}{\|f\|} \right\| + \left\| \frac{g}{\|f\|} - \frac{g}{\|g\|} \right\|$$

$$\leq \frac{1}{\|f\|} \{ \|f+g\| + \|g\| - \|f\| \} \leq \frac{2}{\|f\|} \|f+g\|$$

$$\therefore \|f\| \leq 2\tau^{-1} \|f+g\| \quad \therefore K \leq 2\tau^{-1}$$

$$(i) \leftrightarrow (ii) : 2P = I + iH \text{ より明らか。}$$

$$(ii) \rightarrow (iv) : f \in e^{ib\theta} \mathcal{P}, \quad g \in e^{ia\theta} \overline{\mathcal{P}} \text{ とする。}$$

$$(ii) \text{ より, } \forall t \in \mathbb{R}, \quad t^2 \frac{K^2-1}{K^2} \int_{\mathbb{T}} |f|^2 W \, d\theta + \int_{\mathbb{T}} |g|^2 W \, d\theta + 2t \operatorname{Re} \int_{\mathbb{T}} f \bar{g} W \, d\theta \geq 0$$

この 2 次不等式の判別式 ≤ 0 ゆえ,

$$\left| \int_{\mathbb{T}} f \bar{g} W \, d\theta \right| \leq \frac{\sqrt{K^2-1}}{K} \left\{ \int_{\mathbb{T}} |f|^2 W \, d\theta \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{T}} |g|^2 W \, d\theta \right\}^{\frac{1}{2}}$$

$$(iv) \rightarrow (ii) :$$

$$\|f+g\|_W^2 = \|f\|_W^2 + \|g\|_W^2 + 2 \operatorname{Re} \int_{\mathbb{T}} f \bar{g} W \, d\theta$$

$$(iv) \text{ より, } \begin{aligned} &\geq \|f\|_W^2 + \|g\|_W^2 - 2\rho \|f\|_W \|g\|_W \\ &\geq (1-\rho^2) \|f\|_W^2 \end{aligned}$$

$$(iv) \rightarrow (v) :$$

$$W \in L^1(d\theta) \text{ より, } [\mathcal{P}]_{L^2(W)} \supset H^\infty \text{ ゆえ, (iii) は,}$$

$$\forall f \in e^{ib\theta} H^\infty, \quad \forall g \in e^{ia\theta} \overline{H^\infty} \text{ について成り立つ。}$$

$$\text{この時, } \forall F \in H^\infty, \quad \exists B : \text{Blaschke product, } \exists \Phi \in H^\infty : (|z| < 1) \text{ に zero を持たない。}$$

$$\text{s.t. } F = B\Phi^2 \quad \text{a.e. } \mathbb{T}$$

$f \equiv e^{ib\theta} \Phi$, $g \equiv e^{ia\theta} \overline{B\Phi}$ とおくと,

$$\left| \int_{\mathbb{T}} e^{i(b-a)\theta} F W d\theta \right| = \left| \int_{\mathbb{T}} f \bar{g} W d\theta \right| \leq \rho \|f\|_W \|g\|_W = \rho \int_{\mathbb{T}} |f| W d\theta$$

$$\therefore \left| \int_{\mathbb{T}} e^{i(b-a)\theta} \frac{|\varphi|}{\varphi} (\varphi F) d\theta \right| \leq \rho \int_{\mathbb{T}} |\varphi F| d\theta$$

$\varphi \in H^1$ outer より, $\{\varphi F; F \in H^\infty\} \subseteq H^1$ dense である,

$$\forall G \in H^1, \left| \int_{\mathbb{T}} e^{i(b-a)\theta} \frac{|\varphi|}{\varphi} G d\theta \right| \leq \rho \int_{\mathbb{T}} |G| d\theta$$

Hahn-Banach の双対定理より

$$\left\| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + H^\infty \right\| \leq \rho < 1$$

(v) $\rightarrow$ (iv) :

$$\rho = \left\| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + H^\infty \right\| < 1$$

Hahn-Banach の双対定理より

$$\forall G \in H^1, \left| \int_{\mathbb{T}} e^{i(b-a)\theta} \frac{|\varphi|}{\varphi} G d\theta \right| \leq \rho \int_{\mathbb{T}} |G| d\theta$$

$$\therefore \forall F \in H^\infty, \left| \int_{\mathbb{T}} e^{i(b-a)\theta} F W d\theta \right| \leq \rho \int_{\mathbb{T}} |F| d\theta \quad (\because G = \varphi F)$$

$\therefore \forall f \in e^{ib\theta} \mathcal{P}, \forall g \in e^{ia\theta} \overline{\mathcal{P}}$ について,

$$\left| \int_{\mathbb{T}} f \bar{g} W d\theta \right| \leq \rho \int_{\mathbb{T}} |f \bar{g}| W d\theta \leq \rho \|f\|_W \|g\|_W$$

(v) $\rightarrow$ (vi) : 別証明になっている。

H^∞ の単位球は weak* compact である, $\exists h \in H^\infty$ s.t. $\left\| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + h \right\|_\infty = \rho < 1$

$$\therefore |\arg e^{-i(b-a-1)\theta} \varphi h| \leq \frac{\pi}{2} - \varepsilon \quad \text{a.e.} \quad \text{但し, } \cos \varepsilon = \rho$$

$$\nu(\theta) \equiv \arg e^{-i(b-a-1)\theta} \varphi h \quad \therefore \|\nu\|_\infty \leq \frac{\pi}{2} - \varepsilon$$

Zygmund の定理より, $g(\theta) \equiv e^{i\nu(\theta) - i\nu(\theta)} \in H^1(d\theta)$

$$S(\theta) \equiv e^{i(b-a-1)\theta} \varphi h g \geq 0 \quad \text{a.e.}$$

$e^{i(b-a-1)\theta} S = \varphi h g \in H^{\frac{1}{2}}$ である, $\exists B$: Blaschke product

$\exists \Psi \in H^1 : \{|z| < 1\}$ に zero を持たない。

$$\text{s.t. } e^{i(b-a-1)\theta} S = B \Psi^2 \quad \text{a.e.}$$

$$\therefore e^{-i(b-a-1)\theta} B \Psi^2 = S \geq 0 \quad \text{a.e.}$$

$$\therefore e^{-i(b-a-1)\theta} B \Psi^2 = |\Psi|^2 = \Psi \overline{\Psi} \quad \text{a.e.}$$

$$\Psi \in H^1 \text{ より, } \Psi \neq 0 \quad \text{a.e. } \mathbb{T} \text{ 上で,}$$

$$e^{-i(b-a-1)\theta} B \Psi = \overline{\Psi} \quad \text{a.e.}$$

両辺の Fourier 係数が一致することから, Ψ は高々 $b-a-1$ 次の正則多項式である。

$$\text{この時, } |\varphi h g| = |S| = |\Psi|^2 \quad \text{a.e. } \mathbb{T} \text{ 上で,}$$

$$W = |\varphi| = \frac{|\Psi|^2}{|hg|} = |\Psi|^2 \frac{1}{|h|} e^{-\nu \alpha} = |\Psi|^2 e^{\mu - \nu \alpha}$$

但し, $\mu(\theta) \equiv -\log |h(\theta)|$

$$(iv) \text{ より, } \mu \in L^\infty(d\theta), \|\mu\|_\infty \leq \max \left\{ \log \frac{1}{1-\rho}, \log(1+\rho) \right\}$$

(vi) $\rightarrow$ (v):

$W = |\Psi|^2 e^{\mu + \nu}$, Ψ の zeros は, 全て $\mathbb{T}$ 上にあるように, 取り直しておく。特に, $\Psi \in H^\infty \text{ outer}$ となる。

(iv) と (i) の同値性より, $\mu \equiv 0$ としてよい。

$$W = |\Psi|^2 e^\nu \text{ の時, } \varphi \equiv \Psi^2 e^{\nu - i\nu}, \|\nu\|_\infty \leq \frac{\pi}{2} - \varepsilon \text{ 上で, Zygmond の定理より,}$$

$$\varphi \in H^1 \text{ outer, } |\varphi| = W \quad \text{a.e.}$$

$$\therefore e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} = e^{i(b-a-1)\theta} \frac{e^\nu}{\Psi^2 e^{\nu - i\nu}} = \frac{e^{i(b-a-1)\theta} \overline{\Psi}}{\Psi} e^{i\nu}$$

$$\text{この時 } \frac{e^{i(b-a-1)\theta} \overline{\Psi}}{\Psi} \in N_+ \cap L^\infty = H^\infty$$

$$\begin{aligned} \therefore \left\| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} - \frac{e^{i(b-a-1)\theta} \overline{\Psi}}{\Psi} \cos \|\nu\|_\infty \right\|_\infty \\ = \|e^{i\nu} - \cos \|\nu\|_\infty\|_\infty \leq \sin \|\nu\|_\infty < 1 \end{aligned}$$

■

§ 3. $p=2$ のとき, 2 つの凸集合 $(P, W, (a, b))$ と $(H, W, (a, b))$

[補題 2]

$$W \geq 0 \in L^1(d\theta), \quad a \leq -1 < 0 \leq b$$

$$\text{この時, } \forall U \in (P, W, (a, b)) \text{ について, } \log(W - U) \in L^1(d\theta)$$

証明

$$\int_{\mathbb{T}} |f|^2 U d\theta \leq \int_{\mathbb{T}} |f + g|^2 W d\theta, \quad \forall f \in e^{ib\theta} \mathcal{P}, \quad \forall g \in e^{ia\theta} \overline{\mathcal{P}}$$

$$\therefore \forall t \in \mathbb{R}, \quad t^2 \int_{\mathbb{T}} |f|^2 (W - U) d\theta + 2t \operatorname{Re} \int_{\mathbb{T}} f \overline{g} W d\theta + \int_{\mathbb{T}} |g|^2 W d\theta \geq 0$$

この 2 次方程式の判別式 ≤ 0 上で,

$$\begin{aligned} \left| \int_{\mathbb{T}} f \bar{g} W d\theta \right|^2 &\leq \left\{ \int_{\mathbb{T}} |f|^2 (W-U) d\theta \right\} \left\{ \int_{\mathbb{T}} |g|^2 W d\theta \right\} \\ g &\equiv e^{ia\theta} \\ \left| \int_{\mathbb{T}} e^{-ia\theta} f W d\theta \right|^2 &\leq \left\{ \int_{\mathbb{T}} |f|^2 (W-U) d\theta \right\} \left\{ \int_{\mathbb{T}} W d\theta \right\} \\ \therefore \left| \int_{\mathbb{T}} \frac{W}{W-U} e^{-ia\theta} f (W-U) d\theta \right| &\leq \left\{ \int_{\mathbb{T}} |f|^2 (W-U) d\theta \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{T}} W d\theta \right\}^{\frac{1}{2}} \end{aligned}$$

この時, $\int_{\mathbb{T}} \log (W-U) d\theta = -\infty$ と仮定する。

Szegő の定理より, $\forall f \in L^2(W-U)$ について, 不等式が成り立つ。

$$\begin{aligned} \therefore \int_{\mathbb{T}} \left(\frac{W}{W-U} \right)^2 (W-U) d\theta &\leq \int_{\mathbb{T}} W d\theta \\ \therefore \int_{\mathbb{T}} U d\theta &\leq \int_{\mathbb{T}} \frac{WU}{W-U} d\theta \leq 0 \end{aligned}$$

$$\therefore U \equiv 0 \quad \text{a.e. 仮定より} \quad \int_{\mathbb{T}} \log W d\theta = -\infty$$

これは, $\log W \in L^1(d\theta)$ に矛盾。

$$\therefore \log (W-U) \in L^1(d\theta)$$

■

[命題 3]

$W \geq 0 \in L^1(d\theta) \quad a \leq -1 < 0 \leq b$ この時,

$$(P, W, (a, b)) = \left\{ U \geq 0 ; \exists g \in H^\infty \quad \text{s.t.} \quad U \leq \left(1 - \left| e^{i(b-a-1)\theta} \frac{|\phi|}{\varphi} + g \right|^2 \right) W \quad \text{a.e.} \right\}$$

証明

$\supseteq$: 定理で既に証明した。

$\subseteq$ を示す : $\forall U \in (P, W, (a, b))$ fix.

命題 4 より, $\log (W-U) \in L^1(d\theta)$ ゆえ,

$$\exists \phi \in H^\infty \quad \text{outer} \quad \text{s.t.} \quad |\phi|^4 = \frac{W-U}{W} \quad \text{a.e.}$$

$$\text{一方, } \int_{\mathbb{T}} |f|^2 U d\theta \leq \int_{\mathbb{T}} |f+g|^2 W d\theta, \quad \forall f \in e^{ib\theta} \mathcal{P}, \quad \forall g \in e^{ia\theta} \overline{\mathcal{P}}$$

より, $\left| \int_{\mathbb{T}} f \bar{g} W d\theta \right| \leq \left\{ \int_{\mathbb{T}} |f|^2 (W-U) d\theta \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{T}} |g|^2 W d\theta \right\}^{\frac{1}{2}}$ を得る。…………☆

$\forall F \in \mathcal{P}, \quad \exists B : \text{Blaschke product}, \quad \exists \Phi \in \mathcal{P} : \{|z| < 1\}$ に zeros を持たない。

$$\text{s.t. } F=B\Phi^2 \quad \text{a.e.}$$

$$\text{この時, } \Phi\psi\in[\mathcal{P}]_{L^2(W)}, \text{ 且 } B\Phi\psi^{-1}\in[\mathcal{P}]_{L^2(W-U)}$$

$$\odot \varphi^{\frac{1}{2}}\Phi\psi\in H^2=\left[\varphi^{\frac{1}{2}}\mathcal{P}\right]_{L^2(d\theta)} \quad (\because \text{Beurling の定理による。})$$

$$\therefore \Phi\psi\in[\mathcal{P}]_{L^2(|\varphi|)}=[\mathcal{P}]_{L^2(W)}$$

$$\text{一方, } \varphi^{\frac{1}{2}}\psi B\Phi\psi^{-1}=\varphi^{\frac{1}{2}}B\Phi\in H^2=\left[\varphi^{\frac{1}{2}}\psi\mathcal{P}\right]_{L^2(d\theta)}$$

$$\therefore B\Phi\psi^{-1}\in[\mathcal{P}]_{L^2\left(\left|\varphi^{\frac{1}{2}}\psi\right|^2\right)}=[\mathcal{P}]_{L^2(W-U)}$$

$$\heartsuit \text{えに, } g\equiv e^{ia\theta}\overline{\Phi\psi}\in[e^{ia\theta}\overline{\mathcal{P}}]_{L^2(W)}$$

$$f\equiv e^{ib\theta}B\Phi\psi^{-1}\in[e^{ib\theta}\mathcal{P}]_{L^2(W-U)}$$

$$\therefore \left|\int_{\mathbb{T}} e^{i(b-a)\theta}FWd\theta\right|=\left|\int_{\mathbb{T}} f\overline{g}Wd\theta\right|$$

$$\star \text{より } \leq \left\{\int_{\mathbb{T}} |f|^2(W-U)d\theta\right\}^{\frac{1}{2}}\left\{\int_{\mathbb{T}} |g|^2Wd\theta\right\}^{\frac{1}{2}}$$

$$=\int_{\mathbb{T}} |F|W^{\frac{1}{2}}(W-U)^{\frac{1}{2}}d\theta$$

$$TF\equiv \int_{\mathbb{T}} e^{i(b-a)\theta}FWd\theta, \quad \forall F\in \mathcal{P}$$

$$d\mu(\theta)\equiv W^{\frac{1}{2}}(W-U)^{\frac{1}{2}}d\theta$$

$$|TF|=\left|\int_{\mathbb{T}} e^{i(b-a)\theta}FWd\theta\right|\leq \int_{\mathbb{T}} |F|d\mu(\theta)$$

従って, T は, Hahn-Banach の定理より, ノルムを保って $(L^1(d\mu))^*$ に拡張される。

$$\therefore \exists k\in L^\infty(d\mu) \quad \text{s.t.} \quad \|k\|_\infty\leq 1, \quad \int_{\mathbb{T}} e^{i(b-a)\theta}FWd\theta=\int_{\mathbb{T}} Fkd\mu(\theta)$$

この時 $\log W, \log(W-U)\in L^1(d\theta)$ $\heartsuit$ え, $L^\infty(d\mu)=L^\infty(d\theta)$ となる。

$$\therefore G\equiv kW^{\frac{1}{2}}(W-U)^{\frac{1}{2}}-e^{i(b-a)\theta}W\in e^{i\theta}H^1$$

$$\therefore g\equiv e^{-i\theta}\frac{G}{\varphi}\in N_+\cap L^\infty(d\theta)=H^\infty$$

$$\left|e^{i(b-a-1)\theta}\frac{|\varphi|}{\varphi}+g\right|=|k|\left|\frac{W-U}{W}\right|^{\frac{1}{2}}\leq \left|\frac{W-U}{W}\right|^{\frac{1}{2}} \quad \text{a.e.}$$

$$\therefore U\leq \left(1-\left|e^{i(b-a-1)\theta}\frac{|\varphi|}{\varphi}+g\right|^2\right)W \quad \text{a.e.}$$

■

[注意]

この命題は, 凸集合 $(P, W, (a, b))$ の almost everywhere の意味での極大元は全て,

$$U = \left(1 - \left|e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g\right|^2\right) W \text{ なる形になる事を言っている。}$$

特に, $a = -1, b = 0$ の時は, $U = \left(1 - \left|\frac{|\varphi|}{\varphi} + g\right|^2\right) W$ なる形になり, $\frac{|\varphi|}{\varphi} = \frac{F}{|F|}$ a.e. なる $F \in H^1$ は,

$F = \frac{1}{\varphi}$ に限る事は, Neuwirth and Newman [87] の定理より明らか。

従って, Adamian, Arov and Krein の定理 (Garnett [39] p.160) より, 極大元は全て,

$$U = \left(1 - \left|\frac{|\varphi|}{\varphi} - \frac{2\frac{1}{\varphi}}{G + \frac{1+u}{1-u}}\right|^2\right) W = W - \left|W - \frac{2}{G + \frac{1+u}{1-u}}\right|^2$$

但し, $G(z) \equiv \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \frac{1}{W(\theta)} d\theta, u \in H^\infty, \|u\|_\infty \leq 1$ となる。

[定義]

$W \geq 0, \in L^1(d\theta), a \leq -1 < 0 \leq b$

$$(H, W, (a, b)) \equiv \left\{ U \left| \begin{array}{l} U(\theta) \geq 0 \text{ a.e.} \\ \int_{\mathbb{T}} |Hf|^2 (W - U) d\theta \leq \int_{\mathbb{T}} |f|^2 (W + U) d\theta \\ \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}} \end{array} \right. \right\}$$

[命題 4]

$$(H, W, (a, b)) = \left\{ U \leq W ; \exists g \in H^\infty \text{ s.t. } U \geq \left|e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g\right| W \text{ a.e.} \right\}$$

証明

$\supseteq$ を示す :

$$\forall h \in H^\infty, f \in e^{ib\theta} \mathcal{P}, g \in e^{ia\theta} \overline{\mathcal{P}}$$

$$\operatorname{Re} \int_{\mathbb{T}} f \bar{g} W d\theta \leq \int_{\mathbb{T}} |f \bar{g}| |W + \bar{e}^{i(b-a-1)\theta} \varphi g| d\theta$$

$$\leq \int_{\mathbb{T}} |f \bar{g}| \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g \right| W d\theta$$

$$\leq \frac{1}{2} \int_{\mathbb{T}} (|f|^2 + |g|^2) \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g \right| W d\theta$$

$$\begin{aligned} & \therefore \int_{\mathbb{T}} |f-g|^2 \left(1 - \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g \right| \right) W d\theta \\ & \leq \int_{\mathbb{T}} |f+g|^2 \left(1 + \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g \right| \right) W d\theta \end{aligned}$$

$\subseteq$ を示す :

$$\begin{aligned} & \int_{\mathbb{T}} |f-g|^2 (W-U) d\theta \leq \int_{\mathbb{T}} |f+g|^2 (W+U) d\theta \\ & \therefore \left| \int_{\mathbb{T}} f \bar{g} W d\theta \right| \leq \frac{1}{2} \int_{\mathbb{T}} (|f|^2 + |g|^2) U d\theta \\ & \therefore \left| \int_{\mathbb{T}} e^{i(b-a)\theta} F W d\theta \right| \leq \int_{\mathbb{T}} |F| U d\theta \quad \forall F \in \mathcal{P} \end{aligned}$$

$$TF \equiv \int_{\mathbb{T}} e^{i(b-a)\theta} F W d\theta$$

$$\therefore |TF| \leq \int_{\mathbb{T}} |F| U d\theta$$

Hahn-Banach の定理より, $\mathbb{T}$ を $L^1(U)^*$ に, ノルムを保って拡張できる。

$$\therefore \exists k \in L^\infty(U) \quad \text{s.t.} \quad \|k\|_\infty \leq 1, \quad \int_{\mathbb{T}} e^{i(b-a)\theta} F W d\theta = \int_{\mathbb{T}} F k U d\theta$$

この時, $\log U = \log \frac{1}{2}((W+U) - (W-U))$ ゆえ, 命題 4 より,

$\log U \in L^1(d\theta)$ 。従って, $L^\infty(U) = L^\infty(d\theta)$

$$\therefore G \equiv kU - e^{i(b-a)\theta} W \in e^{i\theta} H^1$$

$$g \equiv e^{-i\theta} \frac{G}{\varphi} \in N_+ \cap L^\infty(d\theta) = H^\infty$$

$$\therefore \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g \right| = \left| e^{i(b-a)\theta} \frac{|\varphi|}{\varphi} + \frac{G}{\varphi} \right|$$

$$= |kU| \frac{1}{W} \leq \frac{U}{W} \quad \text{a.e.}$$

$$\therefore U \geq \left| e^{i(b-a-1)\theta} \frac{|\varphi|}{\varphi} + g \right| \quad \text{a.e.} \quad \blacksquare$$

[注意]

前の命題の注意と同様に, $a=-1$, $b=0$ の時は, 凸集合 $(H, W, (-1, 0))$ の極小元は, $u \in H^\infty$, $\|u\|_\infty \leq 1$ なる u を parameter として, 記述できる。

§ 4. $1 \leq p < \infty$ のとき

今までは Hilbert 空間 $L^2(W)$ の中で考えてきた。

次に, $L^p(W)$ $1 < p < \infty$ の場合を少し考えてみる。

$$(T, W, (a, b))_p \equiv \left\{ U \left| \begin{array}{l} U(\theta) \geq 0 \quad \text{a.e.} \\ \int_{\mathbb{T}} |Tf|^p U \, d\theta \leq \int_{\mathbb{T}} |f|^p W \, d\theta \\ \forall f \in e^{ib\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}} \end{array} \right. \right\}$$

[命題 5] (単位円周 $\mathbb{T}$)

$$1 < p < \infty, \quad W \geq 0, \quad \in L^1(d\theta), \quad a \leq -1 < 0 \leq b, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

次の (i) ~ (iii) は同値である。

$$(i) \quad (P_b, W, (a, b))_p \neq \emptyset$$

$$(ii) \quad \exists \Psi \in \mathcal{P} : b-a-1 \text{ 次 s.t. } \int_{\mathbb{T}} |\Psi|^{p'} W^{-\frac{1}{p-1}} d\theta < \infty$$

$$(iii) \quad \exists \varphi \in H^1 \text{ outer, } |\varphi(\theta)| = W(\theta) \text{ a.e. } \exists g \neq 0 \in H^\infty$$

$$\text{s.t. } \left\| e^{i(b-a-1)\theta} \left(\frac{|\varphi|}{\varphi} \right)^{\frac{1}{p-1}} - g \right\|_\infty \leq 1$$

証明

(i) $\rightarrow$ (ii) :

双対定理より

$$\begin{aligned} 0 < \delta &\equiv \inf \left\{ \left\{ \int_{\mathbb{T}} |e^{ib\theta} + g|^p W \, d\theta \right\}^{\frac{1}{p}} : g \in e^{i(b+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}} \right\} \\ &= \max \left\{ \left| \int_{\mathbb{T}} e^{-ib\theta} h \, W \, d\theta \right| : \int_{\mathbb{T}} |h|^{p'} W \, d\theta = 1, \int_{\mathbb{T}} e^{-in\theta} h \, W \, d\theta = 0, \forall n \notin (a, b] \right\} \\ &= \max \left\{ \left| \widehat{F}(b) \right| : \int_{\mathbb{T}} |F|^{p'} W^{-\frac{1}{p-1}} d\theta = 1, \widehat{F}(n) = 0, \forall n \notin (a, b] \right\} \end{aligned}$$

但し, $F \equiv hW$, Hölder の不等式より, $F \in L^1(d\theta)$

この max を attain する F を F_0 とおく。

$$= \left| \widehat{F_0}(b) \right| = \left\{ \int_{\mathbb{T}} \left| \frac{F_0}{\widehat{F_0}(b)} \right|^{p'} W^{-\frac{1}{p-1}} d\theta \right\}^{-\frac{1}{p'}}$$

$$\therefore \Psi \equiv \frac{F_0}{\widehat{F_0}(b)}, \quad \int_{\mathbb{T}} |\Psi|^{p'} W^{-\frac{1}{p-1}} d\theta = \delta^{-p'} < \infty$$

$\Psi \in \mathcal{P} : b-a-1$ 次

(ii) → (i) :

Hölder の不等式より, $\forall g \in e^{i(b+1)\theta} \mathcal{P} + e^{ia\theta} \overline{\mathcal{P}}$ と $\forall F = \sum_{n=a+1}^b c_n e^{in\theta}$, $C_b = 1$ について,

$$\begin{aligned} & \left\{ \int_{\mathbb{T}} |e^{ib\theta} + g|^p W d\theta \right\}^{\frac{1}{p}} \left\{ \int_{\mathbb{T}} |F|^{p'} W^{-\frac{1}{p-1}} d\theta \right\}^{\frac{1}{p'}} \\ & \geq \left| \int_{\mathbb{T}} (e^{-ib\theta} + \bar{g}) F d\theta \right| = |\widehat{F}(b)| = 1 \\ & \therefore \inf_g \left\{ \int_{\mathbb{T}} |e^{ib\theta} + g|^p W d\theta \right\}^{\frac{1}{p}} \geq \sup_{\substack{F \\ \widehat{F}(b)=1}} \left\{ \int_{\mathbb{T}} |F|^{p'} W^{-\frac{1}{p-1}} d\theta \right\}^{-\frac{1}{p'}} \end{aligned}$$

(ii) より > 0

(iii) → (ii) :

次の事実を使う。(Koosis [64] p.231)

$$\left(\begin{array}{l} u(\theta) : \mathbb{T} \text{ 上の unimodular について, 次の (a), (b) は同値。} \\ (a) \quad \exists g \neq 0 \in H^\infty \quad \text{s.t.} \quad \|u - g\|_\infty \leq 1 \\ (b) \quad \exists k \in H^1 \quad \text{outer} \quad \text{s.t.} \quad u = \frac{k}{|k|} \quad \text{a.e.} \end{array} \right.$$

従って, (iii) より,

$$\exists k \in H^1 \quad \text{outer} \quad \text{s.t.} \quad e^{i(b-a-1)\frac{p'}{2}\theta} \left(\frac{|\varphi|}{\varphi} \right)^{\frac{1}{p-1}} = \frac{k}{|k|} \quad \text{a.e.}$$

$$e^{-i(b-a-1)\theta} \varphi^{\frac{2}{p}} k^{\frac{2}{p'}} \geq 0 \quad \text{a.e.}$$

$$\Psi \equiv \varphi^{\frac{1}{p}} k^{\frac{1}{p'}} \in H^1 \text{ outer}, \quad e^{-i(b-a-1)\theta} \Psi^2 \geq 0 \quad \text{a.e.}$$

$$\therefore e^{-i(b-a-1)\theta} \Psi^2 = |\Psi|^2 = \Psi \overline{\Psi} \quad \text{a.e.}$$

$$\Psi \in H^1 \text{ より, } \Psi \neq 0 \quad \text{a.e.} \quad \text{よ} \searrow \text{え, } e^{-i(b-a-1)\theta} \Psi = \overline{\Psi} \quad \text{a.e.}$$

両辺の Fourier 係数は一致するから,

$$\Psi = C_0 + C_1 e^{i\theta} + \cdots + C_{b-a-1} e^{i(b-a-1)\theta} \text{ となる。}$$

$$\text{この時} \int_{\mathbb{T}} |\Psi|^{p'} W^{-\frac{1}{p-1}} d\theta = \int_{\mathbb{T}} |k| d\theta < \infty$$

(ii) → (iii) :

$$\Psi = \prod_{j=1}^{b-a-1} (e^{i\theta} - a_j), \quad \forall |a_j| = 1 \text{ と, 取り直しておく。}$$

(ii) より容易に, $\log W \in L^1(d\theta)$ を得る。

$$\therefore \exists \varphi \in H^1 \text{ outer} \quad \text{s.t.} \quad |\varphi(\theta)| = W(\theta) \quad \text{a.e.}$$

$$\text{一方} \exists k \in H^1 \text{ outer} \quad \text{s.t.} \quad |k(\theta)| = |\Psi|^{p'} W^{-\frac{1}{p-1}} \quad \text{a.e.}$$

$$\therefore \varphi^{\frac{1}{p-1}} k = \gamma \prod_{j=1}^{b-a-1} (e^{i\theta} - a_j)^{p'}, \quad |\gamma| = 1, \quad \gamma = 1 \text{ としてよい。}$$

$$\begin{aligned} \therefore e^{i(b-a-1)\frac{p'}{2}\theta} \left(\frac{|\varphi|}{\varphi} \right)^{\frac{1}{p-1}} &= e^{i(b-a-1)\frac{p'}{2}\theta} \left(\prod_{j=1}^{b-a-1} \frac{|e^{i\theta} - a_j|^2}{(e^{i\theta} - a_j)} \right)^{\frac{p'}{2}} \frac{k}{|k|} \\ &= \left(\prod_{j=1}^{b-a-1} \frac{e^{-i\theta} - \bar{a}_j}{e^{i\theta} - a_j} e^{i\theta} \right)^{\frac{p'}{2}} \frac{k}{|k|} = C \frac{k}{|k|}, \quad |C|=1 \text{ 定数} \end{aligned}$$

再び前の事実より, (i) を得る。 ■

[注意]

定理 1 の証明と同様にして, 次の (iv), (v) も同値なる事が示される。実際, (i) $\rightarrow$ (iv) $\rightarrow$ (v) $\rightarrow$ (i) となる。

$$(iv) \quad \forall n \geq 0, \quad \exists U_n \in (P_n, W, (a, b))_p \quad \text{s.t.} \quad U_n \neq 0 \quad \text{a.e.}$$

$$(v) \quad \forall E \subseteq \mathcal{P} : \text{有限次元部分空間} \quad \text{s.t.} \quad e^{ib\theta} \in E$$

$$\text{に対して,} \quad \exists U_E \in (P_n, W, (a, b))_p \quad \text{s.t.} \quad U_n \neq 0 \quad \text{a.e.}$$

[命題 6]

$$0 < p < \infty$$

$\mathcal{D} \subseteq \{\text{三角多項式の全体}\}$ 部分空間

$$T : \mathcal{D} \text{ 中の作用素で, } \forall \alpha \in \mathbb{R}, \quad (Tf)_\alpha = Tf_\alpha, \quad \forall f \in \mathcal{D}$$

が成り立つ。但し, $f_\alpha(\theta) \equiv f(\theta + \alpha)$ とする。

この時, 次の (i), (ii) は同値である。

$$(i) \quad \exists W \in L^1(d\theta) \quad \text{s.t.} \quad (T, W, \mathcal{D})_p \neq \{0\}$$

$$(ii) \quad \int_{\mathbb{T}} |Tf|^p d\theta \leq C \int_{\mathbb{T}} |f|^p d\theta, \quad \forall f \in \mathcal{D}$$

証明

(ii) $\rightarrow$ (i) : 明らか。

(i) $\rightarrow$ (ii) :

$$\exists U \neq 0 \quad \text{s.t.} \quad \forall \alpha \in \mathbb{R}, \quad \int_{\mathbb{T}} |Tf_\alpha(\theta)|^p U(\theta) d\theta \leq \int_{\mathbb{T}} |f_\alpha(\theta)|^p W(\theta) d\theta, \quad \forall f \in \mathcal{D}$$

$$Tf_\alpha = (Tf)_\alpha \text{ なる仮定より, } Tf_\alpha(\theta) = (Tf)_\alpha(\theta) = (Tf)(\theta + \alpha) \text{ ゆえ,}$$

$$\forall \alpha \in \mathbb{R} \quad \int_{\mathbb{T}} |(Tf)(\theta + \alpha)|^p U(\theta) d\theta \leq \int_{\mathbb{T}} |f(\theta + \alpha)|^p W(\theta) d\theta, \quad \forall f \in \mathcal{D}$$

$U, W \in L^1(d\theta), Tf, f \in L^\infty(d\theta)$ ゆえ, 両辺とも, $L^\infty(d\alpha)$ に属する。

$$\therefore \int_{\mathbb{T}} \left\{ \int_{\mathbb{T}} |(Tf)(\theta + \alpha)|^p U(\theta) d\theta \right\} d\alpha \leq \int_{\mathbb{T}} \left\{ \int_{\mathbb{T}} |f(\theta + \alpha)|^p W(\theta) d\theta \right\} d\alpha$$

Fubini の定理と Lebesgue 積分の不変性より

$$\left\{ \int_{\mathbb{T}} |(Tf)(\alpha)|^p d\alpha \right\} \left\{ \int_{\mathbb{T}} U(\theta) d\theta \right\} \leq \left\{ \int_{\mathbb{T}} |f(\alpha)|^p d\alpha \right\} \left\{ \int_{\mathbb{T}} W(\theta) d\theta \right\}$$

$$\therefore \int_{\mathbb{T}} |Tf|^p d\theta \leq \frac{\|W\|_1}{\|U\|_1} \int_{\mathbb{T}} |f|^p d\theta, \quad \forall f \in \mathcal{D}$$

[系]

$\Lambda \subset \mathbb{Z}$ 有限集合, $\mathcal{D} \equiv \{f \in \text{三角多項式}; \hat{f}(n) = 0, \quad \forall n \in \Lambda\}$

この時, $\forall W \in L^1(d\theta), (H, W, \mathcal{D})_1 = (P, W, \mathcal{D})_1 = \{0\}$

証明

Duren の本 [30] p.63 より, $\forall n \geq 2, f(\theta) \equiv \sum_{k=n}^{\infty} \frac{\cos k\theta}{\log k} \in L^1(d\theta), Hf(\theta) \notin L^1(d\theta)$ ゆえ,

H は $[e^{in\theta} \mathcal{P} + e^{-in\theta} \overline{\mathcal{P}}]_{L^1(d\theta)}$ の中で, 非有界である。特に, $[\mathcal{D}]_{L^1(d\theta)}$ の中でも, 非有界である。

一方, $2P = I + iH$ ゆえ, P についても同様である。

$(Pf)_\alpha = Pf_\alpha, (Hf)_\alpha = Hf_\alpha$ ゆえ, 上の命題に帰着する。

[命題 7]

$1 < p < \infty, d\nu, d\mu$: 正値有限正則測度

$\exists n$ s.t. $\int_{\mathbb{T}} |pf|^p d\nu \leq \int_{\mathbb{T}} |f|^p d\mu, \quad \forall f \in e^{in\theta} \mathcal{P} + e^{-in\theta} \overline{\mathcal{P}}$

$\Leftrightarrow d\nu$: 絶対連続, 且 $\nu_a \leq \mu_a$ a.e.

証明

Hölder の不等式より, $\frac{1}{p} + \frac{1}{p'} = 1$

$$\left| \int_{\mathbb{T}} Pf d\nu \right| \leq \left(\int_{\mathbb{T}} d\nu \right)^{\frac{1}{p'}} \left(\int_{\mathbb{T}} |Pf|^p d\nu \right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{T}} d\nu \right)^{\frac{1}{p'}} \left(\int_{\mathbb{T}} |f|^p d\mu \right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbb{T}} d\nu \right)^{\frac{1}{p'}} \left(\int_{\mathbb{T}} d\mu \right)^{\frac{1}{p}} \|f\|_\infty$$

$\phi f \equiv \int_{\mathbb{T}} Pf d\nu, \phi$ は $e^{in\theta} \mathcal{P} + e^{-in\theta} \overline{\mathcal{P}}$ 上の $\|\cdot\|_\infty$ による有界線形汎関数ゆえ, Hahn-Banach の定理より, ϕ を $C(\mathbb{T})$ 上の $\exists \Phi$ に拡張できる。この時, Riesz の定理より,

$$\exists dV: \text{正値有限正則測度} \quad \text{s.t.} \quad \Phi f = \begin{cases} \int_{\mathbb{T}} f dV, & f \in C(\mathbb{T}) \\ \phi f, & f \in e^{in\theta} \mathcal{P} + e^{-in\theta} \overline{\mathcal{P}} \end{cases}$$

$\forall k \leq -n, \int_{\mathbb{T}} e^{ik\theta} dV = \phi e^{ik\theta} = 0$ ゆえ, F. and M. Riesz の定理より dV は絶対連続である。同様に,

$\forall k \geq n, \int_{\mathbb{T}} e^{ik\theta} d(\nu - V) = \phi e^{ik\theta} - \Phi e^{ik\theta} = 0$ より, $d(\nu - V)$ も絶対連続ゆえ, $d\nu$ も絶対連続である。

$$\therefore dv = dV_a$$

一方, $\int_{\mathbb{T}} |f|^2 d(\mu - \nu_a) \geq 0$, $\forall f \in e^{in\theta} \mathcal{P}$ ゆえ, μ, ν_a の正則性より, $d(\mu - \nu_a) \geq 0$ となる。

$$\therefore \mu_a \geq \nu_a \quad \text{a.e.}$$

■

§ 5. 実数直線上のとき

[定義]

$\mathbb{R}$: 実軸 (実数全体)

dx : $\mathbb{R}$ 上の Lebesgue 測度

$$W(x) \geq 0, \quad \in L^1(dx)$$

$$\mathcal{P} \equiv \text{span}\{e^{i\lambda x}; \lambda \geq 0 \in \mathbb{R}\}$$

$$\alpha, \beta \in \mathbb{R}, \quad \alpha \leq 0 \leq \beta, \quad \alpha \neq \beta$$

T : 定義域が $\text{span}\{e^{i\lambda x}; \lambda \in \mathbb{R}\}$ の作用素

$$(T, W, (\alpha, \beta)) \equiv \left\{ U \left| \begin{array}{l} U(x) \geq 0 \quad \text{a.e. } \mathbb{R} \\ \int_{\mathbb{R}} |Tf|^2 U \, dx \leq \int_{\mathbb{R}} |f|^2 W \, dx \\ \forall f \in e^{i\beta x} \mathcal{O}f + e^{i\alpha x} \overline{\mathcal{O}f} \end{array} \right. \right\}$$

T として, 次のような作用素を考える。

“ $\widehat{}$ ” は, $\mathbb{R}$ 上の Fourier 変換を表わす。

$$H: \mathbb{R} \text{ 上の Hilbert 変換} \quad \widehat{Hf}(\lambda) = \begin{cases} -i \widehat{f}(\lambda) & \lambda \geq 0 \\ i \widehat{f}(\lambda) & \lambda < 0 \end{cases} \in \mathbb{R}$$

$$P: H^\infty \text{ の中への射影作用素} \quad \widehat{Pf}(\lambda) = \begin{cases} \widehat{f}(\lambda) & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}$$

$$Hf(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(t)}{x-t} dt \quad \text{と書ける。}$$

但し, $\text{p.v.} \int$ は, Cauchy の主値積分を表わす。

[補題 8]

$$\forall \beta > 0, \quad (P, W, (0, \beta)) = \{0\}$$

証明

$$\int_{\mathbb{R}} U dx \leq \int_{\mathbb{R}} |1 - e^{i\lambda x}|^2 W dx, \quad \forall \lambda < 0$$

$$\therefore \int_{\mathbb{R}} U dx \leq \lim_{\lambda \uparrow 0} \int_{\mathbb{R}} |1 - e^{i\lambda x}|^2 W dx = \lim_{\lambda \uparrow 0} \int_{\mathbb{R}} |1 - e^{i\lambda x}|^2 W dx = 0$$

$$\therefore U \equiv 0 \quad \text{a.e.}$$

■

[定理 2]

$$W \geq 0, \in L^1(dx)$$

$\alpha > 0$, $\in \mathbb{R}$ について, 次の (i) ~ (v) は同値である。

$$(i) \|Hf\|_W \leq C \|f\|_W \quad \forall f = \sum_{|\lambda| \geq \alpha} C_n e^{in\theta}, \quad C_n \in \mathbb{C}$$

$$(ii) \|Pf\|_W \leq K \|f\|_W \quad \forall f = \sum_{|\lambda| \geq \alpha} C_n e^{in\theta}, \quad C_n \in \mathbb{C}$$

$$(iii) \sup_{\substack{f = \sum_{|\lambda| \geq \alpha} C_n e^{in\theta}, \\ g = \sum_{|\lambda| < \alpha} C'_n e^{in\theta}}} |(f, g)_W| = \rho < 1$$

$$(iv) \exists \varphi \in H^1 \text{ outer, } |\varphi| = W \text{ a.e. } \mathbb{R}$$

$$\text{s.t. } \left\| e^{2i\alpha x} \frac{|\varphi|}{\varphi} + H^\infty \right\| = \rho < 1$$

$$(v) \exists \Psi : \text{指数型高々 } \alpha \text{ の整関数}$$

$$\exists \mu, \nu \in L^\infty(dx) \text{ s.t. } \|\nu\|_\infty = \frac{\pi}{2} - \varepsilon, \quad W = |\Psi|^2 e^{\mu + \bar{\nu}} \text{ a.e. } \mathbb{R}.$$

証明

(i) $\leftrightarrow$ (ii) $\leftrightarrow$ (iii) $\leftrightarrow$ (iv) の証明は, 命題 1 と同じ。

(vi) $\rightarrow$ (v) :

$$H^\infty \text{ の単位球は weak }^* \text{ compact } \text{ である, } \exists h \in H^\infty \text{ s.t. } \left\| e^{2i\alpha x} \frac{|\varphi|}{\varphi} + h \right\|_\infty = \rho < 1$$

$$\therefore |\arg e^{-2i\alpha x} \varphi h| \leq \frac{\pi}{2} - \varepsilon \text{ a.e. } \cos \varepsilon \leq \rho$$

$$\exists V(\zeta) : \text{harmonic in } (|\zeta| < 1)$$

$$V(e^{i\theta}) = \arg e^{-2i\alpha x} \varphi(x) h(x) \text{ a.e.}$$

$$\text{但し, } e^{i\theta} = \frac{i-x}{i+x}, \quad \zeta = \frac{i-z}{i+z}$$

$$\text{Zygmund の定理より, } e^{\tilde{V}(\zeta) - iV(\zeta)} \in H^1(|\zeta| < 1)$$

$$G(z) \equiv e^{\tilde{V}(\zeta) - iV(\zeta)}$$

$$\therefore -\arg G(x) = V\left(\frac{i-x}{i+x}\right) = \arg e^{-2i\alpha x} \varphi(x) h(x) \text{ a.e.}$$

$$\therefore S(x) \equiv e^{-2i\alpha x} \varphi(x) h(x) G(x) \geq 0 \text{ a.e.}$$

Koosis [62] より, $\exists \Psi : \text{指数型高々 } \alpha \text{ の整関数}$

$$\text{s.t. } S(x) = |\Psi(x)|^2 \text{ a.e. } \mathbb{R}$$

$$\therefore |\Psi(x)|^2 = S(x) = |\varphi(x) h(x) G(x)| \text{ a.e.}$$

$$\therefore W(x) = |\varphi(x)| = \frac{|\Psi(x)|^2}{|h(x) G(x)|} = |\Psi(x)|^2 \frac{1}{|h(x)|} e^{-\tilde{V}(\zeta)}$$

$$\mu(x) \equiv \log \frac{1}{|h(x)|} \in L^\infty(dx), \|u\|_\infty \leq \max \left\{ \log \frac{1}{1-\rho}, \log(1+\rho) \right\}$$

$$\nu(x) \equiv V(e^{i\theta}), \nu(x) \in L^\infty(dx), \|\nu\|_\infty = \frac{\pi}{2} - \varepsilon$$

$$\therefore W(x) = |\Psi(x)|^2 e^{\mu(x) + \tilde{\nu}(x)} \quad \text{a.e.}$$

(v) $\rightarrow$ (iv): Schwarz の不等式より,

$$\int_{\mathbb{R}} \frac{|\Psi(x)|}{\sqrt{1+x^2}} dx \leq \left\{ \int_{\mathbb{R}} \frac{e^{-\mu-\tilde{\nu}}}{1+x^2} dx \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{R}} |\Psi|^2 e^{\mu+\tilde{\nu}} dx \right\}^{\frac{1}{2}} < \infty$$

$$\text{特に } \int_{\mathbb{R}} \frac{|\Psi(x)|}{1+x^2} dx < \infty \quad \text{より} \quad \int_{\mathbb{R}} \frac{\log^+ |\Psi(x)|}{1+x^2} dx < \infty \quad \text{さらに}$$

$$\Psi \text{ は指数型の整関数ゆえ, } \int_{\mathbb{R}} \frac{\|\log \Psi(x)\|}{1+x^2} dx < \infty \quad \text{となる。}$$

$$\therefore \int_{\mathbb{R}} \frac{\log W(x)}{1+x^2} dx = \int_{\mathbb{R}} \frac{\log |\Psi|^2}{1+x^2} dx - \int_{\mathbb{R}} \frac{\log e^{-\mu-\tilde{\nu}}}{1+x^2} dx$$

$$\geq -2 \int_{\mathbb{R}} \frac{\|\log \Psi\|}{1+x^2} dx - \int_{\mathbb{R}} \frac{e^{-\mu-\tilde{\nu}}}{1+x^2} dx > -\infty$$

$$\therefore \exists \varphi \in H^1(dx) \quad \text{outer s.t.} \quad |\varphi(x)| = W(x) \quad \text{a.e.}$$

$e^\mu \in L^\infty(dx)$ ゆえ, $\mu \equiv 0$ としても, 一般性を失わないことは (iv) と (ii) の同値性により, 容易にわかる。

従って, $W = |\Psi|^2 e^{\tilde{\nu}}$ について示す。

$$G \equiv e^{-\tilde{\nu} + i\nu} \quad \text{この時, } \frac{e^{2iax} |\Psi|^2}{\varphi G} \in H^\infty(dx) \quad \text{なる事を, 少しの間認めると,}$$

$$\left| e^{2iax} \frac{|\varphi|}{\varphi} - \frac{e^{2iax} |\Psi|^2}{\varphi G} \cos \|\nu\|_\infty \right| = \left| 1 - \frac{|\Psi|^2}{WG} \cos \|\nu\|_\infty \right| = |1 - e^{-i\nu} \cos \|\nu\|_\infty| = |e^{i\nu} - \cos \|\nu\|_\infty| \leq \cos \varepsilon < 1 \quad \text{a.e.}$$

$$\text{最後に, } \frac{e^{2iax} |\Psi|^2}{\varphi G} \in H^\infty(dx) \quad \text{を示す。}$$

⊙一般に, 指数型 α の整関数 Ψ について,

$$\log |\Psi(z)| \leq \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} \log |\Psi(t)| dt + dy \quad z = x + iy, y > 0.$$

なる事が知られている。

Jensen の不等式より,

$$|e^{iaz} \Psi(z)| \leq \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} |\Psi(t)| dt \quad (y > 0)$$

上半平面において, 左辺は劣調和, 右辺は調和ゆえ,

Garnett の本 [39] p.51 より,

$$\frac{e^{iaz}\Psi(z)}{(z+i)^2} \in H^1, (z+i)^2 \in N_+ \text{ ゆえ, 特に, } e^{iaz}\Psi(z) \in N_+$$

同様に, $e^{iaz}\overline{\Psi(\bar{z})} \in N_+$ となる。

$$\text{一方 } \left| \frac{e^{2iax}|\Psi|^2}{\varphi G} \right| = \frac{|\Psi|^2}{W|G|} = 1 \quad \text{a.e.}$$

$$\therefore \frac{e^{2iax}|\Psi|^2}{\varphi G} \in N_+ \cap L^\infty = H^\infty$$

■

[注意]

補題 8 より, $(P, W, (\alpha, \beta)) \neq \{0\}$ なるためには, $\alpha < 0$ でなければならない。従って, 定理 2 における, $\alpha > 0$ なる仮定は, 強くない。

最後に, $\forall \alpha > 0, (P, W, (-\alpha, \alpha)) \neq \{0\}$ なる $W \geq 0 \in L^1(dx)$ の例を作る。

[例 1]

$W(x) \geq 0, \in L^1(dx)$ 且, $\exists G(x): x \geq 0$ で減少する非負偶関数

$$\text{s.t. (i) } \int_0^\infty \frac{\log G}{1+x^2} dx > -\infty$$

$$\text{(ii) } \int_{-\infty}^\infty \frac{G}{(1+x^2)W} dx > -\infty$$

証明: よく知られているように,

$$\forall \alpha > 0, \exists \Psi: \text{指数型 } \alpha \text{ の整関数 s.t. } \frac{\Psi^2}{G} \in L^\infty(\mathbb{R})$$

$$\therefore \int_{-\infty}^\infty \frac{|\Psi|^2}{(1+x^2)W} dx \leq \left\| \frac{\Psi^2}{G} \right\|_\infty \int_{-\infty}^\infty \frac{G}{(1+x^2)W} dx < \infty$$

よって, 定理 A に帰着する。

■

[例 2] (Koosis [65])

$W(x) \geq 0, \in L^1(dx)$

$$\text{(i) } \int_{-\infty}^\infty \frac{\log W}{1+x^2} dx > -\infty$$

$$\text{(ii) } G(x) \equiv \frac{W(x)W(-x)}{W(x)+W(-x)} \quad \text{は, } x \geq 0 \text{ で減少する非負偶関数}$$

証明

$$\int_0^\infty \frac{\log G}{1+x^2} dx \geq \int_{-\infty}^\infty \log W dx - \int_{-\infty}^\infty \frac{W}{1+x^2} dx > -\infty$$

$$\text{一方, } \int_{-\infty}^\infty \frac{G}{(1+x^2)W} dx = \int_{-\infty}^\infty \frac{W(-x)}{W(x)+W(-x)} \frac{dx}{1+x^2} = \frac{\pi}{2} < \infty$$

よって, 例 1 に帰着する。 ■

[例 3]

$$W(x) \geq 0, \quad \in L^1(dx)$$

$$(i) \quad \int_{-\infty}^\infty \frac{\log W}{1+x^2} dx > -\infty$$

(ii) $W(x)$ は $x \geq 0$ で減少する非負偶関数

証明

例 2 より明らか。 ■

[例 4]

$$W(x) \equiv \left(\frac{1}{1+x^2} \right)^n \quad (n \geq 1) \text{ は, 例 3 の条件をみたしている。}$$

$$\text{実際, } \int_0^\infty \frac{\log \left(\frac{1}{1+x^2} \right)^n}{1+x^2} dx = -n \int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx = -n\pi \log 2 > -\infty$$

[例 5] (Koosis [65])

$$W(x) \geq 0, \quad \in L^1(dx)$$

$\exists E(z)$: 指数型の整関数

$$\text{s.t.} \begin{cases} W(x) = \frac{1}{(x^2+1)(|E(x)|^2+1)} \text{ a.e.} \\ \text{且, } \int_{-\infty}^\infty \frac{\log^+ |E(x)|}{1+x^2} dx < \infty \end{cases}$$

証明

Beurling and Malliavin の定理より,

$\forall \alpha > 0, \exists f_\alpha(z) \equiv 0$: 指数型 α の整関数

s.t. $\Psi_\alpha(x), E(x)f_\alpha(x) \in L^\infty(dx)$

この時,

$$\Psi_\alpha(z) \equiv f_\alpha(z) \left(\frac{\sin \frac{\alpha z}{2}}{z} \right)^2 \text{ は, 指数型高々 } 2\alpha \text{ の整関数}$$

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{|\Psi_{\alpha}(x)|^2}{(x^2+1)W(x)} dx &= \int_{-\infty}^{\infty} (|E(x)|^2+1)|\Psi_{\alpha}(x)|^2 dx \\
 &= \int_{-\infty}^{\infty} (|E(x)|^2+1)|f_{\alpha}(x)|^2 \left(\frac{\sin \frac{\alpha x}{2}}{x} \right)^2 dx \\
 &\leq (\|E^2 f_{\alpha}\|_{\infty} + \|f_{\alpha}\|_{\infty}^2) \int_{-\infty}^{\infty} \left(\frac{\sin \frac{\alpha x}{2}}{x} \right)^2 dx \\
 &= (\|E f_{\alpha}\|_{\infty}^2 + \|f_{\alpha}\|_{\infty}^2) \cdot \frac{\pi}{2} \cdot \alpha < \infty
 \end{aligned}$$

■

REFERENCES

- [1] V. Adamian, D. Arov, and M. Krein, *Beskoniecznyye gankeliovy matritsy i obobshchennyye zadachi Karateodori-Feiera i I. Shura*, Funktsionalnyi Analiz i iego Prilozheniia, **2**: (1968), 1–17.
- [2] N. Akhiezer, *Lektsii po teorii approksimatsii*, second augmented edition, Nauka, Moscow, 1965.
- [3] T. Ando, *Projections in Krein spaces*, Linear Algebra Appl. **431** (2009), 2346–2358.
- [4] T. Ando, *Linear Operators on Krein Spaces*, Lecture Note, Hokkaido University, Sapporo, Japan, 1979.
- [5] T. Ando, *Contractive projections in L^p spaces*, Pacific J. Math. **17** (1966), 391–405.
- [6] T. Ando, *Unbounded or bounded idempotent operators in Hilbert space*, Linear Algebra Appl. **438** (2011), 3769–3775.
- [7] R. Arocena, *A refinement of the Helson-Szegő theorem and the determination of the extremal measures*, Studia Math. **71** (1981), 203–221.
- [8] R. Arocena, and M. Cotlar, *A generalized Herglotz-Bochner theorem and L^2 -weighted inequalities with finite measures*, Conference on harmonic analysis in Honor of Antoni Zygmund (Chicago, Ill, 1981), *Wadsworth Math. Ser.*, pp. 258–269, Wadsworth, Belmont, 1983.
- [9] R. Arocena, M. Cotlar, and C. Sadosky, *Weighted inequalities in L^2 and lifting properties*, *Advances in Math. Suppl. Studies* 7A, 95–128, Academic Press, New York, 1981.
- [10] A. Beurling and P. Malliavin, *On Fourier transforms of measures with compact support*, Acta Math., **107** (1962), 201–302.
- [11] E. Bishop and R. Phelps, *The support functionals of a convex set*, Amer. Math. Soc. Proc. Symposia in Puro Math., VII, *Convexity*, (1963), pp. 27–35.
- [12] R. Boas, *Entire functions*, Academic Press New York, 1954.
- [13] A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, Akademie-Verlag, Berlin, and Springer-Verlag, 1990.
- [14] A. Böttcher and I. M. Spitkovsky, *A gentle guide to the basis of two projections theory*, Linear Algebra Appl. **432** (2010), 1412–1439.
- [15] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102.

- [16] D. Buckholtz, *Inverting the difference of Hilbert space projections*, Amer. Math. Monthly **104** (1997), 60–61.
- [17] C. Burnap and I. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl. **337** (2008), 686–694.
- [18] L. Carleson and P. Jones, *Weighted norm inequalities and a theorem of Koosis*, Institut Mittag-Leffler, report no. 2 (1981).
- [19] M. Chō, T. Nakazi and T. Yamazaki, *Hyponormal operators and two-isometry*, Far East J. of Mathematical Sciences **49** (2011), 111–119.
- [20] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
- [21] M. Cotlar, A unified theory of Hilbert transforms and ergodic theorems, Rev. Mat. Cuyana **1** (1955), 105–167.
- [22] M. Cotlar and C. Sadosky, *On the Helson-Szegő theorem and a related class of modified Toeplitz kernels*, pp. 383–407, *Harmonic analysis in Euclidean spaces* (Williamstown, MA, 1978), Part 1, eds. G. Weiss and S. Wainger, Proc. Symp. Pure Math. 35, Amer. Math. Soc., Providence, 1979.
- [23] M. Cotlar and C. Sadosky, *Toeplitz liftings of Hankel forms*, pp. 22–43, *Function spaces and applications* (Lund, 1986), Lect. Notes Math. 1302, Springer-Verlag, Berlin and New York, 1988.
- [24] M. Cotlar and C. Sadosky, *Weakly positive matrix measures, generalized Toeplitz forms, and their applications to Hankel and Hilbert transform operators*, pp. 93–120, *Operator Theory: Adv. and Appl.* (Basel, Birkhäuser), vol. 58, 1992.
- [25] M. Cotlar and C. Sadosky, *Revisiting almost orthogonality and eigen expansions*, *Function spaces, interpolation theory and related topics*, Lund, 2000 (de Gruyter, Berlin, 2002), 249–271.
- [26] C. C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), 809–812.
- [27] M. Dominguez, *Interpolation and prediction problems for connected compact abelian groups*, Integral Equations Operator Theory **40** (2001), 212–230.
- [28] M. Dominguez, *Weighted inequalities for the Hilbert transform and the adjoint operator in the continuous case*, Studia Math. **95** (1990), 229–236.
- [29] R. G. Douglas, *Banach algebra techniques in operator theory (2nd ed.)*, Springer-Verlag, New York, Berlin, 1998.
- [30] P. Duren: *Theory of H^p spaces*. Academic Press, New York 1970.
- [31] V. B. Dybin and S. M. Grudsky, *Introduction to the theory of Toeplitz operators with infinite index*, Birkhäuser-Verlag, Basel, 2002.
- [32] I. Feldman, N. Ya. Krupnik and A. Markus, *On the norm of polynomials of two adjoint projections*, Integral Equations Operator Theory **14** (1991), 69–90.
- [33] F. Forelli: *The Marcel Riesz theorem on conjugate functions*. Trans. Amer. Math. Soc. **106** (1963), 369–390.
- [34] M. Fujii and Y. Nakatsu, *On subclasses of hyponormal operators*, Proc. Japan Acad., Ser. A. **51** (1975), 243–246.
- [35] T. Furuta, *Invitation to Linear Operators from Matrices to Bounded Linear Operators on a Hilbert Space*. Taylor & Francis, London, 2001.
- [36] B. G. Gabdulkhayev, *Optimal approximations of solutions of linear problems* (Kazansk. Gos. Univ., Kazan, 1980) [in Russian].
- [37] F. D. Gakhov, *Boundary-value problems* (Nauka, Moscow, 1977) [in Russian].
- [38] J. Garnett: *Two remarks on interpolation by bounded analytic functions*. “Banach spaces of analytic functions.” Lecture notes in Math. **604**. Springer, Berlin (1977), 32–40.

- [39] J. Garnett, *Bounded analytic functions* (Revised First Edition), Graduate Texts in Math., Springer, Berlin, 2006.
- [40] I. Gohberg and N. Ya. Krupnik, *One-dimensional linear singular integral equations*. "Shtiintsa", Kishinev, 1973 (Russian); English transl.: Vol. I and II, Birkhäuser-Verlag, Basel, 1992.
- [41] F. D. Gakhov, *Boundary value problems* [in Russian], Fizmatgiz, Moscow (1963).
- [42] C. Gu, *Algebraic properties of Cauchy singular integral operators on the unit circle*, Taiwanese J. Math. **20** (2016), 161–189.
- [43] C. Gu, I. S. Hwang, D. Kang and W. Y. Lee, *Normal singular Cauchy integral operators with operator-valued symbols*, J. Math. Anal. Appl. **447** (2017), 289–308.
- [44] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.
- [45] P. R. Halmos, *A Hilbert space problem book*, 2nd ed., Springer-Verlag, 1982.
- [46] H. Helson and D. Sarason, Past and future, Math. Scand. **21** (1967), 5–16.
- [47] H. Helson and G. Szegő, *A problem in prediction theory*, Annali di Mat. Bologna (IV), vol. **51** (1960), 107–138.
- [48] K. Hoffman, *Banach spaces of analytic functions*, Englewood Cliffs, 1962.
- [49] R. Hunt, B. Muckenhoupt and R. Wheeden: *Weighted norm inequalities for the conjugate function and Hilbert transform*. Trans. Amer. Math. Soc. **176** (1973) 227–251.
- [50] I. S. Hwang and W. Y. Lee, *Subnormal Toeplitz operators and the kernels of their self-commutators*, J. Math. Anal. Appl. **361** (2010), 270–275.
- [51] I. S. Hwang and W. Y. Lee, *Hyponormal Toeplitz operators with rational symbols*, J. Oper. Theory **56** (2006), 47–58.
- [52] T. Ito and T. K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34** (1972), 157–164.
- [53] Z. Jabłoński and J. Stochel, *Unbounded 2-hyperexpansive operators*, Proc. Edinburgh Math. Soc. **44** (2001), 613–629.
- [54] J.-P. Kahane, *Another theorem on bounded analytic functions*, Proc. Amer. Math. Soc., **18** (1967), pp. 827–831.
- [55] J.-P. Kahane, *Sur les fonctions moyenne-périodiques bornées*, Ann. Inst. Fourier Grenoble **7** (1957), 293–314.
- [56] B. V. Khvedelidze, "The method of Cauchy-type integrals in the discontinuous boundary value problems of the theory of holomorphic functions of a complex variable," in: Contemporary Problems of Mathematics [in Russian], VINITI, Moscow, 1975, 7, pp. 5–162 (Itogi Nauki i Tekhniki).
- [57] Y. Kim, E. Ko, J. Lee and T. Nakazi, *Hyponormality of singular Cauchy integral operators with matrix-valued symbols*, preprint.
- [58] E. Ko, I. E. Lee and T. Nakazi, *On the dilation of truncated Toeplitz operators II*, Preprint.
- [59] E. Ko, I. E. Lee and T. Nakazi, *Hyponormality of the dilation of truncated Toeplitz operators*, in preparation.
- [60] P. Koosis, *Sur la non-totalité de certaines suites d'exponentielles*, Ann. Sci. École Norm. Sup. (3), **75** (1958), 125–152.
- [61] P. Koosis, *Interior compact spaces of functions on a half-line*, Comm. Pure. Appl. Math. **10** (1957), 583–615.
- [62] P. Koosis: *Moyennes quadratiques pondérées de transformées de Hilbert et fonctions de type exponentiel*. C. R. Acad. Sci. Paris **276** (1973), 1201–1204.
- [63] P. Koosis: *Moyennes quadratiques pondérées de fonctions périodiques et de leurs conjuguées harmoniques*, C. R. Acad. Sci. Paris **291** (1980), 255–257.

- [64] P. Koosis, *Introduction to H^p spaces (2nd ed.)*, Cambridge Univ. Press, 1998.
- [65] P. Koosis, *Weighted quadratic means of Hilbert transforms*, Duke Math. J. **38** (1971), 609–634.
- [66] N. Ya. Krupnik, *Banach algebras with symbol and singular integral operators*, “Shtiintsa”, Kishinev, 1984 (Russian); English transl.: Birkhäuser-Verlag, Basel, 1987.
- [67] N. Ya. Krupnik and I. E. Verbitsky, *Exact constants in the theorem of K. I. Babenko and B. V. Khvedelidze on the boundedness of singular operators* (Russian), *Soobshzh. AN Gruz. SSR* **85** (1977), no. 1, 21–24.
- [68] P. Lax, *Remarks on the preceding paper*, Comm. Pure Appl. Math., **10** (1957), 617–622.
- [69] I. K. Lifanov, *The Method of Singular Integral Equations and the Numerical Experiment* (“Yanus”, Moscow, 1995) [in Russian].
- [70] B. A. Lotto, *Range inclusion of Toeplitz and Hankel operators*, J. Operator Theory **24** (1990), 17–22.
- [71] R. A. Martínez-Avendaño, Rosenthal P.: *An introduction to operators on the Hardy-Hilbert space*. Springer, Berlin (2007).
- [72] S. G. Mikhlin, and S. Prössdorf, *Singular integral operators*, Springer-Verlag, Berlin and New York, 1986.
- [73] J. Moeller and P. Frederickson, *A density theorem for lacunary Fourier series*, Bull. Amer. Math. Soc., **72** (1966), 82–86.
- [74] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*. Trans. Amer. Math. Soc. **165** (1972), 207–226.
- [75] B. Muckenhoupt, *Weighted norm inequalities for classical operators*. Proc. Symposia in Pure Math. **35**, Part I (1979), 69–83.
- [76] N. I. Muskhelishvili, *Singular integral equations* (Nauka, Moscow, 1968) [in Russian].
- [77] T. Nakazi, *Range inclusion of two same type concrete operators*, preprint.
- [78] T. Nakazi, *Norm inequality of $AP + BQ$ for selfadjoint projections P and Q with $PQ = 0$* , J. Math. Ineq. **7** (2013), 513–516.
- [79] T. Nakazi, *Hyponormal singular integral operators with Cauchy kernel on L^2* , Commun. Korean Math. Soc. **33** (2018), 787–798.
- [80] T. Nakazi, *Exposed points and extremal problems in H^1* , J. Funct. Anal. **53** (1983), 224–230.
- [81] T. Nakazi, *Weighted norm inequalities and uniform algebras*, Proc. Amer. Math. Soc. **103** (1988), 507–512.
- [82] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), 753–767.
- [83] T. Nakazi, and T. Yamamoto, *A Lifting theorem and uniform algebras*, Trans. Amer. Math. Soc. **305** (1988), 79–94.
- [84] T. Nakazi, and T. Yamamoto, *Some singular integral operators and Helson-Szegő measures*, J. Funct. Anal. **88** (1990), 366–384.
- [85] T. Nakazi and T. Yamamoto, *Norms of some singular integral operators*, J. Operator. Th. **40** (1998), 187–207.
- [86] T. Nakazi and T. Yamamoto, *Normal singular integral operators with Cauchy kernel on L^2* , Integr. Equ. Oper. Th. **78** (2014), 233–248.
- [87] J. Neuwirth and D. J. Newman: *Positive $H^{\frac{1}{2}}$ functions are constant*. Proc. Amer. Math. Soc. **18** (1967), 958.
- [88] N. K. Nikolski, *Operators, functions, and systems: An Easy Reading*. Vol. 1, Amer. Math. Soc., Providence, 2002.
- [89] N. K. Nikolskii, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986.
- [90] A. V. Ozhegova, *“The uniform approximations of solutions to weakly singular integral equations of*

- the first kind*,” Candidate’s Dissertation in Mathematics and Physics (Kazan, 1996).
- [91] A. V. Ozhegova and L. E. Valiulova, “*The uniform approximation of solutions to singular integral equations of the first kind by projective methods*,” in Proceedings of the First International Research and Practice Conference “Scientific Potential to the World’ 2004,” 2004 (Nauka i Osvita, Dnepropetrovsk, 2004), **31**, p. 64.
 - [92] R. Phelps, *Extreme points in function algebras*, Duke Math. J. **32** (1965), 267–278.
 - [93] R. Phelps, *Weak* support points of convex sets in E^** , Israel J. Math. **2** (1964), 177–182.
 - [94] S. C. Power, *Hankel Operators on Hilbert space*, Pitman, Boston, Mass., 1982.
 - [95] S. Prösdorf, *Some classes of singular equations* [Russian translation], Mir, Moscow (1979).
 - [96] S. Richter, *A representation theorem for cyclic analytic two isometries*, Trans. Amer. Math. Soc. **328** (1991), 325–349.
 - [97] R. Rochberg, *Toeplitz operators on weighted H^p spaces*, Indiana Univ. Math. J. **26** (1977), 291–298.
 - [98] W. Rogosinski and H. S. Shapiro, *On certain extremum problems for analytic functions*, Acta Math., **90** (1953), 287–318.
 - [99] Rubio de Francia, *Boundedness of maximal functions and singular integrals in weighted L^p spaces*. Proc. Amer. Math. Soc. **83** (1981), 673–679.
 - [100] C. Sadosky, *Some applications of majorized Toeplitz kernels*, pp. 581–626, *Topics in Modern Harmonic Analysis*, Proc. Seminar Torino and Milano (May-June 1982), Vol. II Inst. Naz. Alta Matematica F. Severi, Roma, 1983.
 - [101] C. Sadosky, *The mathematical contributions of Mischa Cotlar since 1955*, Analysis and Partial Differential Equations, pp. 715–742, Dekker, New York, 1990.
 - [102] C. Sadosky, *Liftings of kernels shift-invariant in scattering theory*, pp. 303–336, Holomorphic spaces, eds. Sh. Axler, J. McCarthy and D. Sarason, MSRI Publications 33, Cambridge Univ. Press, 1998.
 - [103] D. Sarason, *Generalized interpolation in H^∞* , Trans. Amer. Math. Soc. **127** (1967), 179–203.
 - [104] D. Sarason: *An addendum to “Past and future.”* Math. Scand. **30** (1972), 62–64.
 - [105] D. Sarason, *Function theory on the unit circle*, Virginia Polytechnic Institute and State Univ., Blacksburg, VA, 1979.
 - [106] D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices, **1** (2007), 419–526.
 - [107] E. Sawyer, *Two weight norm inequalities for certain maximal and integral operators*. Lecture Notes in Math. **908**. Springer, Berlin (1981), 102–127.
 - [108] S. M. Shimorin, *Wold-type decompositions and wandering subspaces of operators close to isometries*, J. reine angew. Math. **531** (2001), 147–189.
 - [109] Y. Sone and T. Yoshino, *Remark on the range inclusions of Toeplitz and Hankel operators*, Proc. Japan Acad., Ser. A. **71** (1995), 168–170.
 - [110] I. M. Spitkovsky, *On partial indices of continuous matrix-valued functions*, Soviet Math. Doklady **17** (1976), 1155–1159.
 - [111] T. Yamamoto, *On the generalization of the theorem of Helson and Szegő*, Hokkaido Math. J. **14** (1985), 1–11.
 - [112] T. Yamamoto, *On weighted norm inequalities in L^2 on the unit circle*, Journal of Hokkai-Gakuen University **52** (1985), 13–19.
 - [113] T. Yamamoto, *Boundedness of some singular integral operators in weighted L^2 spaces*, J. Operator Theory **32** (1994), 243–254.
 - [114] T. Yamamoto, *Majorization of singular integral operators with Cauchy kernel on L^2* , Ann. Funct. Anal. **5** (2014), 101–108.
 - [115] N. Young, *An introduction to Hilbert space*. Cambridge Univ. Press, 1988.
 - [116] A. Zygmund, *Trigonometric series*, I, II, (Third Edition) Cambridge Univ. Press, 2002.