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# Normality of Some Singular Integral Type Operators on the Hilbert Space Dedicated to Professor Tsuyoshi Ando on his 90th birthday

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#### Abstract

We study the normality of  $\Phi P + \Psi Q$  when P is selfadjoint and Q = I - P. These results are applied to three special cases.

# §1. Introduction

Let *L* be a Hilbert space and *H* a closed subspace of *L*. Let *P* denote the orthogonal projection from *L* to *H* and *I* denote the identity operator on *L*. Suppose  $L = H \oplus K$  and Q = I - P where *H* = *PL*, *K* = *QL*, and *B*(*L*) denotes the set of all bounded linear operators on *L*.

An operator X in  $\mathcal{B}(L)$  is called a normal operator when  $X^*X - XX^* = 0$ . For  $\Phi$  and  $\Psi$  in  $\mathcal{B}(L)$ ,  $S_{\Phi,\Psi} = \Phi P + \Psi Q$  is called a singular integral type operator. In this paper, we are interested in when  $S_{\Phi,\Psi}$  is a normal operator. In order to study it, we need two kinds of operators. The first ones are  $T_{\Phi} = P\Phi P$  and  $\tilde{T}_{\Phi} = Q\Phi Q$ . The second ones are  $H_{\Phi} = Q\Phi P$  and  $\tilde{H}_{\Phi} = P\Phi Q$ . Then  $T_{\Phi}^* = T_{\Phi^*}$ ,  $\tilde{T}_{\Phi}^* = \tilde{T}_{\Phi^*}$  and  $H_{\Phi}^* = \tilde{H}_{\Phi^*}$ . It is clear that  $PS_{\Phi,\Psi}P = T_{\Phi}$ ,  $QS_{\Phi,\Psi}Q = \tilde{T}_{\Psi}$ ,  $QS_{\Phi,\Psi}P = H_{\Phi}$  and  $PS_{\Phi,\Psi}Q = \tilde{H}_{\Psi}$ .

### § 2. Necessary and sufficient conditions

In this section, we give a necessary and sufficient condition for normal  $S_{\Phi,\Psi}$ . As a result, we study normal  $S_{\Phi,\Psi}$  when  $\Phi - \lambda \Psi = cI$  where  $\lambda$ ,  $c \in \mathbb{C}$  and  $|\lambda| = 1$ .

**Lemma 2–1.** Let  $\Phi$  and  $\Psi$  be in B(L). Then

$$\begin{split} S_{\Phi,\Psi}^* & S_{\Phi,\Psi} - S_{\Phi,\Psi} S_{\Phi,\Psi}^* = \\ & \begin{bmatrix} (T_{\Phi^*\Phi} - T_{\Phi\Phi^*}) + (\tilde{H}_{\Phi}H_{\Phi^*} - \tilde{H}_{\Psi}H_{\Psi^*}) & (\tilde{H}_{\Phi^*\Psi} - \tilde{H}_{\Psi\Phi^*}) + (\tilde{H}_{\Psi}\tilde{T}_{(\Phi-\Psi)^*} - T_{\Phi-\Psi}\tilde{H}_{\Phi^*}] \\ & (H_{\Psi^*\Phi} - H_{\Phi\Psi^*}) + (\tilde{T}_{\Phi-\Psi}H_{\Psi^*} - H_{\Phi}T_{(\Phi-\Psi)^*}) & (\tilde{T}_{\Psi^*\Psi} - \tilde{T}_{\Psi\Psi^*}) + (H_{\Psi}\tilde{H}_{\Psi^*} - H_{\Phi}\tilde{H}_{\Phi^*}) \end{bmatrix} \end{split}$$

**Theorem 2–1.** Let  $\Phi$  and  $\Psi$  be in B(L). Then  $S_{\Phi,\Psi}$  is normal if and only if

- (1)  $T_{\Phi^*\Phi} = T_{\Phi\Phi^*}, \ \tilde{T}_{\Psi^*\Psi} = \tilde{T}_{\Psi\Psi^*} \text{ and } H_{\Psi^*\Phi} = H_{\Phi\Psi^*},$
- (2)  $\tilde{H}_{\Phi}H_{\Phi^*} = \tilde{H}_{\Psi}H_{\Psi^*}$  and  $H_{\Phi}\tilde{H}_{\Phi^*} = H_{\Psi}\tilde{H}_{\Psi^*}$ ,

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(3)  $T_{\Phi-\Psi}H_{\Psi^*} = H_{\Phi}T_{(\Phi-\Psi)^*}.$ 

**Corollary 2–1.** Suppose  $\Phi$  and  $\Psi$  are normal with  $\Psi^* \Phi = \Phi \Psi^*$ . Then  $S_{\Phi,\Psi}$  is normal if and only if

- (1)  $\tilde{H}_{\Phi}H_{\Phi^*} = \tilde{H}_{\Psi}H_{\Psi^*}$  and  $H_{\Phi}\tilde{H}_{\Phi^*} = H_{\Psi}\tilde{H}_{\Psi^*}$ ,
- (2)  $T_{\Phi-\Psi}H_{\Psi^*} = H_{\Phi}T_{(\Phi-\Psi)^*}.$

**Corollary 2–2.** Let  $\Phi - \lambda \Psi = cI$  and  $\lambda$ ,  $c \in \mathbb{C}$  with  $|\lambda| = 1$ . Then  $S_{\Phi,\Psi}$  is normal if and only if

- (1)  $T_{\Phi^*\Phi} = T_{\Phi\Phi^*}, \ \tilde{T}_{\Phi^*\Phi} = \tilde{T}_{\Phi\Phi^*} \text{ and } H_{\Phi^*\Phi} = H_{\Phi\Phi^*}$
- (2)  $\overline{c}H_{\Phi} = (1 \overline{\lambda}) (T_{\Phi}H_{\Phi^*} + H_{\Phi\Phi^*}).$

**Corollary 2–3.** Let  $\Phi - \Psi = cI$  and  $c \in \mathbb{C}$ . Then  $S_{\Phi,\Psi}$  is normal if and only if

- (1)  $T_{\Phi^*\Phi} = T_{\Phi\Phi^*}, \ \tilde{T}_{\Phi^*\Phi} = \tilde{T}_{\Phi\Phi^*} \text{ and } H_{\Phi^*\Phi} = H_{\Phi\Phi^*}$
- (2)  $\overline{c}H_{\Phi} = 0$

Now we will give necessary and sufficient conditions for selfadjoint  $S_{\Phi,\Psi}$  and nonnegative  $S_{\Phi,\Psi}$ . These are easy to show.

**Theorem 2–2.** Let  $\Phi$  and  $\Psi$  be in B(L). Then  $S_{\Phi,\Psi}$  is selfadjoint if and only if  $T_{\Phi-\Phi^*} = \tilde{T}_{\Psi-\Psi^*} = 0$ and  $H_{\Phi-\Psi^*} = 0$ .

**Theorem 2–3.** Let  $\Phi$  and  $\Psi$  be in B(L). Then  $S_{\Phi,\Psi}$  is nonnegative if and only if

- (1)  $T_{\Phi} \ge 0$  and  $\tilde{T}_{\Psi} \ge 0$ , and  $H_{\Phi-\Psi^*} = 0$ .
- (2)  $|\langle H_{\Phi}f, g \rangle|^2 \leq ||T_{\Phi}f|| ||\tilde{T}_{\Psi}g|| \ (f \in H, g \in K).$

#### § 3. Two sufficient conditions

In this section two typical sufficient conditions are givn. Suppose  $\Phi$ ,  $\Psi F$  are in B(L), and a, b and  $\lambda$  are in B(L). We assume  $\Phi \neq \Psi$ .

**Theorem 3–1.** Suppose  $\Phi = aI + F + \frac{a-b}{\overline{a}-\overline{b}}F^*$  and  $\Psi = bI + F + \frac{a-b}{\overline{a}-\overline{b}}F^*$ . If  $FH \subset H$ 

then  $S_{\Phi,\Psi}$  is normal.

**Theorem 3-2.** Suppose  $\Phi = \lambda aF + b$  and  $\Psi = aF + b$  where  $\lambda$ , a and b are in  $\mathbb{C}$  with  $|\lambda| = 1$  and  $\lambda \neq 1$ , and F is a unitary operator. Then  $S_{\Phi,\Psi}$  is normal.

**Theorem 3–3.** If  $\Phi = \Phi^*$ ,  $\Psi = \Psi^*$  and  $\Phi - \Psi = cI$  for some c in  $\mathbb{C}$  then  $S_{\Phi,\Psi}$  is selfadjoint. **Theorem 3–4.** If  $\Phi \ge 0$ ,  $\Psi \ge 0$  and  $\Phi - \Psi = cI$  for some c in  $\mathbb{C}$  then  $S_{\Phi,\Psi}$  is nonnegative.

## § 4. Necessary conditions

In this section few typical necessary conditions are given. Suppose  $\Phi$ ,  $\Psi$ , f, G and g are elements in B(L), and a, b, c and  $\lambda$  are in  $\mathbb{C}$ .

**Theorem 4–1.** Let  $\Phi - \lambda \Psi = cI$  and  $|\lambda| = 1$ . If  $S_{\Phi,\Psi}$  is normal then  $(\Phi^* \Phi - \Phi \Phi^*) H \subset K$  and

 $\Gamma H \subseteq H$  where  $\Gamma = (\lambda - 1)\Psi\Psi^* + c\Psi^* - \lambda \overline{c}\Psi$ .

**Corollary 4–1.** Let  $\lambda = 1$ . If  $S_{\Phi,\Psi}$  is normal, then  $(\Phi^*\Phi - \Phi\Phi^*) H \subset K$  and  $(c\Phi^* - \overline{c}\Phi) H \subset H$ . **Corollary 4–2.** Let  $\lambda \neq 1$ . If  $S_{\Phi,\Psi}$  is normal, then  $(\Phi^*\Phi - \Phi\Phi^*) H \subset K$  and  $(GG^*) H \subset H$ .

where

$$\Phi = \lambda G + \frac{c}{1-\lambda} I \text{ and } \Psi = G + \frac{c}{1-\lambda} I.$$

**Theorem 4–2.** If  $S_{\Phi,\Psi}$  is selfadjoint then  $(\Phi - \Phi^*) H \subset L$ ,  $(\Psi - \Psi^*) H \subset L$  and  $(\Phi - \Psi^*) H \subset H$ .

**Theorem 4–3.** Suppose  $S_{\phi,\Psi}$  is nonnegative. Then the following hold.

- (1)  $(\Phi \Phi^*) H \subset L, (\Psi \Psi^*) H \subset L \text{ and } (\Phi \Psi^*) H \subset H.$
- (2)  $T_{\Phi} \geq 0$  and  $T_{\Psi} \geq 0$
- (3)  $|\langle (\Phi \Psi^*) f, g \rangle|^2 \leq ||T_{\Phi} f|| ||\tilde{T}_{\Psi}g|| \ (f \in H, g \in L).$

## § 5. Special case I

Let A be a uniform algera on a compact Hausdorff space X. Let m be a representing measure on X for a nonzero complex homomorphism  $\tau$  on A. The abstract Hardy space  $H^p$  is the closure of A in  $L^p$  for  $1 \le p < \infty$  and  $H^{\infty}$  is defined by  $H^2 \cap L^{\infty}$ . We assume that  $H^{\infty} = \{F \in L^{\infty} : FH^2 \subset H^2\}$ =  $\{F \in L^{\infty} : \overline{F}K^2 \subset K^2\}$  where  $K^2 = L^2 \ominus H^2$ . Then  $L^2 = H^2 \oplus K^2$ . Put  $H_0^2 = \{F \in H^2 : \int Fdm = 0\}$ . For  $\phi$  and  $\phi$  in  $L^{\infty}$ ,  $\Phi = M_{\phi}$  and  $\Psi = M_{\phi}$ . We will write  $S_{\phi,\Psi} = S_{\phi,\phi}$ .

**Theorem 5–1.** Let  $\phi - \lambda \psi = c$  where  $\lambda$ ,  $c \in \mathbb{C}$  and  $|\lambda| = 1$ . If  $S_{\phi,\phi}$  is normal then  $(\lambda - 1)|\psi|^2 + c\overline{\psi} - \lambda \overline{c}\psi$  is constant.

**Corollary 5–1.** If  $\lambda = 1$  then  $c\overline{\phi} - \overline{c}\phi$  is real constant.

**Corollary 5–2.** If  $\lambda \neq 1$  then  $S_{\phi,\phi}$  is normal if and only if

$$\phi = \lambda aF + b$$
 and  $\psi = aF + b$ 

where a,  $b \in \mathbb{C}, F \in L^{\circ}$  and  $b = \frac{c}{1-\lambda}$  , |F| = 1

**Theorem 5–2.**  $S_{\phi,\phi}$  is selfadjoint if and only if  $\phi = \overline{\phi}$ ,  $\psi = \overline{\psi}$  and  $\phi - \psi = c$  for some c in  $\mathbb{C}$ .

**Theorem 5–3.** Suppose  $|H^2|$  is dense in  $|L^2|$ . Then  $S_{\phi,\phi}$  is nonnegative if and only if  $\phi \ge 0$ ,  $\phi \ge 0$ and  $\phi - \phi = c$  for some constant c.

## § 6. Special case II

Let q be an inner function, that is, q is a function in  $H^{\infty}$  and |q| = 1 a.e. Let  $H = H^2 \ominus qH^2$ ,  $K = qH^2 \oplus (L^2 \ominus H^2)$ .

For  $\phi$  in  $L^{\infty}$ ,  $\Phi = M_{\phi}$ ,  $T_{\Phi} = T_{\phi}$  and  $S_{\Phi,\Psi} = S_{\phi,\phi}$  where  $\Psi = M_{\phi}$  and  $\phi$  in  $L^{\infty}$ .

**Theorem 6–1.** Let  $\phi - \lambda \phi = c$  where  $\lambda, c \in \mathbb{C}$  and  $|\lambda| = 1$ . If  $S_{\phi,\phi}$  is normal then  $\gamma (H^2 \ominus qH^2)$ 

 $\subset H^2 \ominus qH^2 \text{ where } \gamma = (\lambda - 1) |\psi|^2 + c\overline{\psi} - \lambda \overline{c}\psi. \text{ If } \int q \, dm = 0 \text{ then } \gamma \text{ is constant.}$ 

**Corollary 6–1.** If  $\lambda = 1$  and  $S_{\phi,\phi}$  is normal then  $(c\overline{\phi} - \overline{c}\phi)$   $(H^2 \ominus qH^2) \subset H^2 \ominus qH^2$ .

If  $\int q dm = 0$  then  $c\overline{\phi} - \overline{c}\phi$  is real constant.

**Corollary 6–2.** If  $\lambda \neq 1$  and  $S_{\phi,\phi}$  is normal then

$$\phi = \lambda G + b$$
 and  $\phi = G + b$ 

where  $b = c / (1 - \lambda)$  and  $|G|^2 (H^2 \ominus qH^2) \subseteq H^2 \ominus qH^2$ . If  $\int q dm = 0$  then G = aF where |F| = 1 and  $a \in \mathbb{C}$ .

**Theorem 6–2.** Suppose  $\int q dm = 0$ . Let  $\phi$  and  $\psi$  be in  $L^{\infty}$ . Then  $S_{\phi,\phi}$  is selfadjoint if and only if  $\phi - \overline{\phi}$  belongs to  $qH^2 + \overline{qH^2}$ ,  $\psi = \overline{\psi}$  and  $(\phi - \psi) (H^2 \ominus qH^2) \subseteq H^2 \ominus qH^2$ .

**Theorem 6-3.** Let  $|H^2|$  be dense in  $|L^2|$ . Suppose that  $\int q dm = 0$  and if  $g(H^2 \ominus qH^2) \subset H^2$  $\ominus qH^2$  then g is constant when g is in  $H^{\circ}$ . Then  $S_{\phi,\phi}$  is nonnegative if and only if  $\phi = \overline{\phi}, \phi \ge 0, \phi$  $-\phi$  is constant c, and for f in  $H^2 \ominus qH^2$  and  $g \in qH^2 + K^2$ 

$$\int (\psi + c) \mid f \mid^2 dm \ge 0$$

and

$$\int \psi f \,\overline{g} \, dm \Big|^2 \leq \int (\psi + c) |f|^2 \, dm \int \psi |g|^2 \, dm.$$

#### § 7. Special case III

Let  $L = H^2$ ,  $H = H^2 \ominus qH^2$  (or  $H = qH^2$ ) and  $K = qH^2$  (or  $K = H^2 \ominus qH^2$ , respectively). We assume  $\Phi = T_{\phi}$  and  $\Psi = T_{\phi}$  for  $\phi$  and  $\psi$  in  $L^{\infty}$ . We write  $S_{\phi,\phi} = S_{\phi,\Psi}$ .

**Theorem 7–1.** Suppose  $H = H^2 \ominus qH^2$  and  $K = qH^2$  or  $H = qH^2$  and  $K = H^2 \ominus qH^2$ . Then  $S_{\phi,\phi}$  is normal if and only if

- (1)  $(T^*_{\phi}T_{\phi} T_{\phi}T^*_{\phi}) H \subseteq K, (T^*_{\phi}T_{\phi} T_{\phi}T^*_{\phi}) L \subseteq H \text{ and } (T^*_{\phi}T_{\phi} T_{\phi}T^*_{\phi}) H \subseteq H$
- (2)  $(T_{\phi}P_{H}T_{\phi}^{*} T_{\phi}P_{H}T_{\phi}^{*}) H \subseteq K$  and  $(T_{\phi}P_{H}T_{\phi}^{*} T_{\phi}P_{H}T_{\phi}^{*}) K \subseteq H$ .
- (3)  $(T_{\phi-\phi}P_KT_{\phi}^* T_{\phi}P_HT_{\phi-\phi}^*) H \subseteq H.$

**Corollary 7–1.** Let  $\phi - \lambda \psi = c$  when  $\lambda$ ,  $c \in \mathbb{C}$  and  $|\lambda| = 1$ . Suppose  $H = H^2 \odot qH^2$  and  $K = qH^2$  or  $H = qH^2$  and  $K = H^2 \odot qH^2$ . Then  $S_{\phi,\phi}$  is normal if and only if

- $(1) \quad T_{\phi}^* T_{\phi} = T_{\phi} T_{\phi}^*$
- (2)  $\overline{c} T_{\phi} H \subset H$
- (3) (2  $(\lambda 1)$   $T_{\phi} T_{\phi}^* \lambda \overline{c} T_{\phi}$ )  $H \subseteq H$ .

**Corollary 7–2.** If  $\lambda = 1$  and  $S_{\phi,\phi}$  is normal then  $T_{\gamma}(qH^2) \subset qH^2$  where  $\gamma = c\overline{\psi} - \overline{c}\psi$ . If A is a weak\*-Dirichlet algebra then  $i\gamma$  is real constant and  $T_{\phi}$  is normal.

**Theorem 7–2.** Suppose  $\int q dm = 0$ . Let  $\phi$  and  $\psi$  be in  $L^{\infty}$ .

(1) Let  $H = H^2 \ominus qH^2$  and  $K = qH^2$ . Then  $S_{\phi,\phi}$  is selfadjoint if and only if  $T_{(\phi-\overline{\phi})} (H^2 \ominus qH^2)$  $\subset qH^2, T_{(\phi-\overline{\phi})} (qH^2) \subset H^2 \ominus qH^2$  and  $T_{(\phi-\overline{\phi})} (H^2 \ominus qH^2) \subset H^2 \ominus qH^2$ .

(2) Let  $H = qH^2$  and  $K = H^2 \ominus qH^2$ . Then  $S_{\phi,\phi}$  is selfadjoint if and only if  $T_{(\phi-\overline{\phi})}qH^2 \subset H^2$  $\ominus qH^2$  and  $T_{(\phi-\overline{\phi})}qH^2 \subset qH^2$ .

- (3) If  $T_{(\phi-\overline{\phi})}(H^2 \ominus qH^2) \subseteq qH^2$  then  $\phi \overline{\phi}$  belongs to  $qH^2 + \overline{qH^2} + N_c^2$ .
- (4) If  $T_{\phi-\overline{\phi}} qH^2 \subseteq H^2 \ominus qH^2$  then  $\psi \overline{\psi}$  belongs to  $N_c^2$

**Corollary 7–3.** Let A be a weak\*-Dirichlet algebra. Suppose  $\int q dm = 0$ .

(1) Let  $H = H^2 \ominus qH^2$  and  $K = qH^2$ . Then  $S_{\phi,\phi}$  is selfadjoint if and only if  $\phi - \overline{\phi}$  belongs to  $qH^2 + \overline{qH^2}, \phi = \overline{\phi}$  and  $\phi - \overline{\phi}$  is constant.

(2) Let  $H = qH^2$  and  $K = H^2 \ominus qH^2$ . Then  $S_{\phi,\phi}$  is selfadjoint if and only if  $\phi = \overline{\phi}, \phi - \overline{\psi}$  belongs to  $qH^2 + \overline{qH^2}$  and  $\phi - \overline{\psi}$  is constant.

**Theorem 7–3.** Suppose  $H = H^2 \ominus qH^2$  and  $K = qH^2$  or  $H = qH^2$  and  $K = H^2 \ominus qH^2$ . Then  $S_{\phi,\phi}$  is nonnegative if and only if for f in H and q in K,  $\int (T_{\phi}f) \overline{f} dm \ge 0$ ,  $\int (T_{\phi}g) \overline{g} dm \ge 0$  and  $\left| \int (T_{\phi}f) \overline{g} dm \right|^2 \le \int (T_{\phi}f) \overline{f} dm \int (T_{\phi}g) \overline{g} dm$ .

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