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Normality of Some Singular Integral Type Operators on the Hilbert Space
Dedicated to Professor Tsuyoshi Ando on his 90th birthday

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Abstract

We study the normality of $\Phi P + \Psi Q$ when P is selfadjoint and $Q = I - P$. These results are applied to three special cases.

§ 1. Introduction

Let L be a Hilbert space and H a closed subspace of L. Let P denote the orthogonal projection from L to H and I denote the identity operator on L. Suppose $L = H \oplus K$ and $Q = I - P$ where $H = PL$, $K = QL$, and $\boldsymbol{B}(L)$ denotes the set of all bounded linear operators on L.

An operator X in $\boldsymbol{B}(L)$ is called a normal operator when $X^*X - XX^* = 0$. For Φ and Ψ in $\boldsymbol{B}(L)$, $S_{\Phi,\Psi} = \Phi P + \Psi Q$ is called a singular integral type operator. In this paper, we are interested in when $S_{\Phi,\Psi}$ is a normal operator. In order to study it , we need two kinds of operators. The first ones are $T_\Phi = P\Phi P$ and $\tilde{T}_\Phi = Q\Phi Q$. The second ones are $H_\Phi = Q\Phi P$ and $\tilde{H}_\Phi = P\Phi Q$. Then $T_\Phi^* = T_{\Phi^*}$, $\tilde{T}_\Phi^* = \tilde{T}_{\Phi^*}$ and $H_\Phi^* = \tilde{H}_{\Phi^*}$. It is clear that $PS_{\Phi,\Psi}P = T_\Phi$, $QS_{\Phi,\Psi}Q = \tilde{T}_\Psi$, $QS_{\Phi,\Psi}P = H_\Phi$ and $PS_{\Phi,\Psi}Q = \tilde{H}_\Psi$.

§ 2. Necessary and sufficient conditions

In this section, we give a necessary and sufficient condition for normal $S_{\Phi,\Psi}$. As a result, we study normal $S_{\Phi,\Psi}$ when $\Phi - \lambda\Psi = cI$ where λ, $c \in \mathbb{C}$ and $|\lambda| = 1$.

Lemma 2–1. Let Φ and Ψ be in $\boldsymbol{B}(L)$. Then

$$S_{\Phi,\Psi}^* S_{\Phi,\Psi} - S_{\Phi,\Psi}S_{\Phi,\Psi}^* = \begin{bmatrix} (T_{\Phi^*\Phi} - T_{\Phi\Phi^*}) + (\tilde{H}_\Phi H_{\Phi^*} - \tilde{H}_\Psi H_{\Psi^*}) & (\tilde{H}_{\Phi^*\Psi} - \tilde{H}_{\Psi\Phi^*}) + (\tilde{H}_\Psi \tilde{T}_{(\Phi-\Psi)^*} - T_{\Phi-\Psi}\tilde{H}_{\Phi^*}) \\ (H_{\Psi^*\Phi} - H_{\Phi\Psi^*}) + (\tilde{T}_{\Phi-\Psi}H_{\Psi^*} - H_\Phi T_{(\Phi-\Psi)^*}) & (\tilde{T}_{\Psi^*\Psi} - \tilde{T}_{\Psi\Psi^*}) + (H_\Psi\tilde{H}_{\Psi^*} - H_\Phi\tilde{H}_{\Phi^*}) \end{bmatrix}$$

Theorem 2–1. Let Φ and Ψ be in $\boldsymbol{B}(L)$. Then $S_{\Phi,\Psi}$ is normal if and only if

(1) $T_{\Phi^*\Phi} = T_{\Phi\Phi^*}$, $\tilde{T}_{\Psi^*\Psi} = \tilde{T}_{\Psi\Psi^*}$ and $H_{\Psi^*\Phi} = H_{\Phi\Psi^*}$,

(2) $\tilde{H}_\Phi H_{\Phi^*} = \tilde{H}_\Psi H_{\Psi^*}$ and $H_\Phi \tilde{H}_{\Phi^*} = H_\Psi \tilde{H}_{\Psi^*}$,

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(3) $\tilde{T}_{\Phi-\Psi}H_{\Psi^*} = H_\Phi T_{(\Phi-\Psi)^*}$.

Corollary 2–1. Suppose Φ and Ψ are normal with $\Psi^*\Phi = \Phi\Psi^*$. Then $S_{\Phi,\Psi}$ is normal if and only if

(1) $\tilde{H}_\Phi H_{\Phi^*} = \tilde{H}_\Psi H_{\Psi^*}$ and $H_\Phi \tilde{H}_{\Phi^*} = H_\Psi \tilde{H}_{\Psi^*}$,

(2) $\tilde{T}_{\Phi-\Psi}H_{\Psi^*} = H_\Phi T_{(\Phi-\Psi)^*}$.

Corollary 2–2. Let $\Phi - \lambda\Psi = cI$ and $\lambda,\ c \in \mathbb{C}$ with $|\lambda| = 1$. Then $S_{\Phi,\Psi}$ is normal if and only if

(1) $T_{\Phi^*\Phi} = T_{\Phi\Phi^*}$, $\tilde{T}_{\Phi^*\Phi} = \tilde{T}_{\Phi\Phi^*}$ and $H_{\Phi^*\Phi} = H_{\Phi\Phi^*}$

(2) $\bar{c}H_\Phi = (1 - \bar{\lambda})\,(T_\Phi H_{\Phi^*} + H_{\Phi\Phi^*})$.

Corollary 2–3. Let $\Phi - \Psi = cI$ and $c \in \mathbb{C}$. Then $S_{\Phi,\Psi}$ is normal if and only if

(1) $T_{\Phi^*\Phi} = T_{\Phi\Phi^*}$, $\tilde{T}_{\Phi^*\Phi} = \tilde{T}_{\Phi\Phi^*}$ and $H_{\Phi^*\Phi} = H_{\Phi\Phi^*}$

(2) $\bar{c}H_\Phi = 0$

Now we will give necessary and sufficient conditions for selfadjoint $S_{\Phi,\Psi}$ and nonnegative $S_{\Phi,\Psi}$. These are easy to show.

Theorem 2–2. Let Φ and Ψ be in $\boldsymbol{B}(L)$. Then $S_{\Phi,\Psi}$ is selfadjoint if and only if $T_{\Phi-\Phi^*} = \tilde{T}_{\Psi-\Psi^*} = 0$ and $H_{\Phi-\Psi^*} = 0$.

Theorem 2–3. Let Φ and Ψ be in $\boldsymbol{B}(L)$. Then $S_{\Phi,\Psi}$ is nonnegative if and only if

(1) $T_\Phi \geq 0$ and $\tilde{T}_\Psi \geq 0$, and $H_{\Phi-\Psi^*} = 0$.

(2) $|\langle H_\Phi f,\ g\rangle|^2 \leq \|T_\Phi f\|\ \|\tilde{T}_\Psi g\|$ $(f \in H,\ g \in K)$.

§ 3. Two sufficient conditions

In this section two typical sufficient conditions are givn. Suppose Φ, Ψ F are in $\boldsymbol{B}(L)$, and a, b and λ are in $\boldsymbol{B}(L)$. We assume $\Phi \neq \Psi$.

Theorem 3–1. Suppose $\Phi = aI + F + \dfrac{a-b}{\bar{a}-\bar{b}}F^*$ and $\Psi = bI + F + \dfrac{a-b}{\bar{a}-\bar{b}}F^*$. If $FH \subset H$ then $S_{\Phi,\Psi}$ is normal.

Theorem 3–2. Suppose $\Phi = \lambda aF + b$ and $\Psi = aF + b$ where λ, a and b are in $\mathbb{C}$ with $|\lambda| = 1$ and $\lambda \neq 1$, and F is a unitary operator. Then $S_{\Phi,\Psi}$ is normal.

Theorem 3–3. If $\Phi = \Phi^*$, $\Psi = \Psi^*$ and $\Phi - \Psi = cI$ for some c in $\mathbb{C}$ then $S_{\Phi,\Psi}$ is selfadjoint.

Theorem 3–4. If $\Phi \geq 0$, $\Psi \geq 0$ and $\Phi - \Psi = cI$ for some c in $\mathbb{C}$ then $S_{\Phi,\Psi}$ is nonnegative.

§ 4. Necessary conditions

In this section few typical necessary conditions are given. Suppose Φ, Ψ, f, G and g are elements in $\boldsymbol{B}(L)$, and a, b, c and λ are in $\mathbb{C}$.

Theorem 4–1. Let $\Phi - \lambda\Psi = cI$ and $|\lambda| = 1$. If $S_{\Phi,\Psi}$ is normal then $(\Phi^*\Phi - \Phi\Phi^*)\ H \subset K$ and

$\Gamma H \subset H$ where $\Gamma = (\lambda - 1)\Psi\Psi^* + c\Psi^* - \lambda\bar{c}\Psi$.

Corollary 4–1. Let $\lambda = 1$. If $S_{\Phi,\Psi}$ is normal, then $(\Phi^*\Phi - \Phi\Phi^*)\, H \subset K$ and $(c\Phi^* - \bar{c}\Phi)\, H \subset H$.

Corollary 4–2. Let $\lambda \neq 1$. If $S_{\Phi,\Psi}$ is normal, then $(\Phi^*\Phi - \Phi\Phi^*)\, H \subset K$ and $(GG^*)\, H \subset H$ where

$$\Phi = \lambda G + \frac{c}{1-\lambda} I \text{ and } \Psi = G + \frac{c}{1-\lambda} I.$$

Theorem 4–2. If $S_{\Phi,\Psi}$ is selfadjoint then $(\Phi - \Phi^*)\, H \subset L$, $(\Psi - \Psi^*)\, H \subset L$ and $(\Phi - \Psi^*)\, H \subset H$.

Theorem 4–3. Suppose $S_{\Phi,\Psi}$ is nonnegative. Then the following hold.

(1) $(\Phi - \Phi^*)\, H \subset L$, $(\Psi - \Psi^*)\, H \subset L$ and $(\Phi - \Psi^*)\, H \subset H$.

(2) $T_\Phi \geq 0$ and $T_\Psi \geq 0$

(3) $|\langle(\Phi - \Psi^*)\, f, g\rangle|^2 \leq \|T_\Phi f\| \, \|\tilde{T}_\Psi g\|$ $(f \in H, g \in L)$.

§ 5. Special case I

Let A be a uniform algera on a compact Hausdorff space X. Let m be a representing measure on X for a nonzero complex homomorphism τ on A. The abstract Hardy space H^p is the closure of A in L^p for $1 \leq p < \infty$ and H^∞ is defined by $H^2 \cap L^\infty$. We assume that $H^\infty = \{F \in L^\infty : FH^2 \subset H^2\} = \{F \in L^\infty : \bar{F}K^2 \subset K^2\}$ where $K^2 = L^2 \ominus H^2$. Then $L^2 = H^2 \oplus K^2$. Put $H_0^2 = \{F \in H^2 : \int F dm = 0\}$. For ϕ and ψ in L^∞, $\Phi = M_\phi$ and $\Psi = M_\psi$. We will write $S_{\Phi,\Psi} = S_{\phi,\psi}$.

Theorem 5–1. Let $\phi - \lambda\psi = c$ where λ, $c \in \mathbb{C}$ and $|\lambda| = 1$. If $S_{\phi,\psi}$ is normal then $(\lambda - 1)|\psi|^2 + c\bar{\psi} - \lambda\bar{c}\psi$ is constant.

Corollary 5–1. If $\lambda = 1$ then $c\bar{\phi} - \bar{c}\phi$ is real constant.

Corollary 5–2. If $\lambda \neq 1$ then $S_{\phi,\psi}$ is normal if and only if

$$\phi = \lambda aF + b \text{ and } \psi = aF + b$$

where $a, b \in \mathbb{C}$, $F \in L^\infty$ and $b = \dfrac{c}{1-\lambda}$, $|F| = 1$

Theorem 5–2. $S_{\phi,\psi}$ is selfadjoint if and only if $\phi = \bar{\phi}$, $\psi = \bar{\psi}$ and $\phi - \psi = c$ for some c in $\mathbb{C}$.

Theorem 5–3. Suppose $|H^2|$ is dense in $|L^2|$. Then $S_{\phi,\psi}$ is nonnegative if and only if $\phi \geq 0$, $\psi \geq 0$ and $\phi - \psi = c$ for some constant c.

§ 6. Special case II

Let q be an inner function, that is, q is a function in H^∞ and $|q| = 1$ *a.e.* Let $H = H^2 \ominus qH^2$, $K = qH^2 \oplus (L^2 \ominus H^2)$.

For ϕ in L^∞, $\Phi = M_\phi$, $T_\Phi = T_\phi$ and $S_{\Phi,\Psi} = S_{\phi,\psi}$ where $\Psi = M_\psi$ and ψ in L^∞.

Theorem 6–1. Let $\phi - \lambda\psi = c$ where $\lambda, c \in \mathbb{C}$ and $|\lambda| = 1$. If $S_{\phi,\psi}$ is normal then $\gamma(H^2 \ominus qH^2) \subset H^2 \ominus qH^2$ where $\gamma = (\lambda - 1)|\psi|^2 + c\overline{\psi} - \lambda\overline{c}\psi$. If $\int q dm = 0$ then γ is constant.

Corollary 6–1. If $\lambda = 1$ and $S_{\phi,\psi}$ is normal then $(c\overline{\phi} - \overline{c}\phi)(H^2 \ominus qH^2) \subset H^2 \ominus qH^2$. If $\int q dm = 0$ then $c\overline{\phi} - \overline{c}\phi$ is real constant.

Corollary 6–2. If $\lambda \neq 1$ and $S_{\phi,\psi}$ is normal then

$$\phi = \lambda G + b \text{ and } \psi = G + b$$

where $b = c/(1-\lambda)$ and $|G|^2(H^2 \ominus qH^2) \subseteq H^2 \ominus qH^2$. If $\int q dm = 0$ then $G = aF$ where $|F| = 1$ and $a \in \mathbb{C}$.

Theorem 6–2. Suppose $\int q dm = 0$. Let ϕ and ψ be in L^∞. Then $S_{\phi,\psi}$ is selfadjoint if and only if $\phi - \overline{\phi}$ belongs to $qH^2 + \overline{q}\overline{H^2}$, $\psi = \overline{\psi}$ and $(\phi - \psi)(H^2 \ominus qH^2) \subseteq H^2 \ominus qH^2$.

Theorem 6–3. Let $|H^2|$ be dense in $|L^2|$. Suppose that $\int q dm = 0$ and if $g(H^2 \ominus qH^2) \subset H^2 \ominus qH^2$ then g is constant when g is in H^∞. Then $S_{\phi,\psi}$ is nonnegative if and only if $\phi = \overline{\phi}$, $\psi \geq 0$, $\phi - \psi$ is constant c, and for f in $H^2 \ominus qH^2$ and $g \in qH^2 + K^2$

$$\int (\psi + c)\,|f|^2\, dm \geq 0$$

and

$$\left|\int \psi f \overline{g} dm\right|^2 \leq \int (\psi + c)|f|^2 dm \int \psi |g|^2 dm.$$

§ 7. Special case III

Let $L = H^2$, $H = H^2 \ominus qH^2$ (or $H = qH^2$) and $K = qH^2$ (or $K = H^2 \ominus qH^2$, respectively). We assume $\Phi = T_\phi$ and $\Psi = T_\psi$ for ϕ and ψ in L^∞. We write $S_{\phi,\psi} = S_{\Phi,\Psi}$.

Theorem 7–1. Suppose $H = H^2 \ominus qH^2$ and $K = qH^2$ or $H = qH^2$ and $K = H^2 \ominus qH^2$. Then $S_{\phi,\psi}$ is normal if and only if

(1) $(T_\phi^* T_\phi - T_\phi T_\phi^*) H \subset K$, $(T_\psi^* T_\psi - T_\psi T_\psi^*) L \subset H$ and $(T_\psi^* T_\phi - T_\phi T_\psi^*) H \subset H$

(2) $(T_\psi P_H T_\phi^* - T_\phi P_H T_\psi^*) H \subset K$ and $(T_\psi P_H T_\phi^* - T_\phi P_H T_\psi^*) K \subset H$.

(3) $(T_{\phi-\psi} P_K T_\psi^* - T_\phi P_H T_{\phi-\psi}^*) H \subset H$.

Corollary 7–1. Let $\phi - \lambda\psi = c$ when $\lambda, c \in \mathbb{C}$ and $|\lambda| = 1$. Suppose $H = H^2 \ominus qH^2$ and $K = qH^2$ or $H = qH^2$ and $K = H^2 \ominus qH^2$. Then $S_{\phi,\psi}$ is normal if and only if

(1) $T_\psi^* T_\phi = T_\phi T_\psi^*$

(2) $\overline{c} T_\psi H \subset H$

(3) $(2(\lambda - 1)\, T_\psi T_\psi^* - \lambda \overline{c} T_\psi) H \subset H$.

Corollary 7–2. If $\lambda = 1$ and $S_{\phi,\psi}$ is normal then $T_\gamma(qH^2) \subset qH^2$ where $\gamma = c\overline{\psi} - \overline{c}\psi$. If A is a weak$*$-Dirichlet algebra then $i\gamma$ is real constant and T_ϕ is normal.

Theorem 7–2. Suppose $\int q\,dm = 0$. Let ϕ and ψ be in L^∞.

(1) Let $H = H^2 \ominus qH^2$ and $K = qH^2$. Then $S_{\phi,\psi}$ is selfadjoint if and only if $T_{(\phi-\overline{\phi})}(H^2 \ominus qH^2) \subset qH^2$, $T_{(\psi-\overline{\psi})}(qH^2) \subset H^2 \ominus qH^2$ and $T_{(\phi-\overline{\phi})}(H^2 \ominus qH^2) \subset H^2 \ominus qH^2$.

(2) Let $H = qH^2$ and $K = H^2 \ominus qH^2$. Then $S_{\phi,\psi}$ is selfadjoint if and only if $T_{(\phi-\overline{\phi})}qH^2 \subset H^2 \ominus qH^2$ and $T_{(\phi-\overline{\phi})}\,qH^2 \subset qH^2$.

(3) If $T_{(\phi-\overline{\phi})}(H^2 \ominus qH^2) \subset qH^2$ then $\phi - \overline{\phi}$ belongs to $qH^2 + \overline{qH^2} + N_c^2$.

(4) If $T_{\psi-\overline{\psi}}\,qH^2 \subset H^2 \ominus qH^2$ then $\psi - \overline{\psi}$ belongs to N_c^2

Corollary 7–3. Let A be a weak$*$-Dirichlet algebra. Suppose $\int q\,dm = 0$.

(1) Let $H = H^2 \ominus qH^2$ and $K = qH^2$. Then $S_{\phi,\psi}$ is selfadjoint if and only if $\phi - \overline{\phi}$ belongs to $qH^2 + \overline{qH^2}$, $\psi = \overline{\psi}$ and $\phi - \overline{\psi}$ is constant.

(2) Let $H = qH^2$ and $K = H^2 \ominus qH^2$. Then $S_{\phi,\psi}$ is selfadjoint if and only if $\phi = \overline{\phi}$, $\psi - \overline{\psi}$ belongs to $qH^2 + \overline{qH^2}$ and $\phi - \overline{\psi}$ is constant.

Theorem 7–3. Suppose $H = H^2 \ominus qH^2$ and $K = qH^2$ or $H = qH^2$ and $K = H^2 \ominus qH^2$. Then $S_{\phi,\psi}$ is nonnegative if and only if for f in H and q in K, $\int (T_\phi f)\,\overline{f}\,dm \geq 0$, $\int (T_\psi g)\,\overline{g}\,dm \geq 0$ and

$$\left|\int (T_\phi f)\overline{g}\,dm\right|^2 \leq \int (T_\phi f)\,\overline{f}\,dm \int (T_\psi g)\overline{g}\,dm.$$

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