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Iwanami Studies in Advanced Mathematics

Hilbert Spaces of Analytic Functions

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Translated by Takanori YAMAMOTO

Introduction

This book is a commentary aimed at helping fourth-year undergraduate students in mathematics understand the space of holomorphic functions. A holomorphic function on the open unit disc in the complex plane can be expressed as a Taylor series there. Its radius of convergence is at least 1. The purpose of this book is to study a space where an inner product is defined in the linear space of such functions, and which becomes complete with respect to the norm determined by that inner product, namely a Hilbert space. Examples of Hilbert spaces of holomorphic functions include Hardy spaces, Bergman spaces, and Dirichlet spaces, all of which are named after people. These spaces are sometimes referred to by the same names even when they do not necessarily form Hilbert spaces, but the subject of this book is the case of Hilbert spaces. However, cases where they do not form Hilbert spaces are used as tools for that research. Here, the space of holomorphic functions is limited to the Hilbert space of holomorphic functions on the unit disc of one variable. Instead of the unit disc of a plane, we can consider more general planar domains or n-dimensional complex domains as the Hilbert space of holomorphic functions. However, the position of this book is to provide an easy-to-understand explanation of the Hilbert space of holomorphic functions on the unit disc of a plane, which is the simplest but has the most profound results. This book deals with a field that has made great progress from the first half of the 20th century to the present day in the 21st century, and continues to make progress. To go into a little more detail, a holomorphic function on an open unit disk in the complex plane can be expanded by Taylor expansion, and the radius of convergence is 1 or more. The purpose of this book is to study Hilbert spaces, which are linear spaces of such complex functions whose inner products are defined and which are complete with respect to the norm determined by these inner products. There have been no books written to date that focus exclusively on Hilbert spaces, which are spaces of holomorphic functions. In that sense, this book is a new attempt. In the study of Hilbert spaces in this book, results on Banach spaces of holomorphic functions are often used. Since the subject of this book is Hilbert spaces in Part I, I would like first-time readers to finish Part I without going into the proofs of the theorems on Banach spaces in Part II that are cited. After that, I would like you to read the proofs in Part II for a deeper understanding. This book deals with Hilbert spaces with general reproducing kernels, which are very typical examples of Hilbert spaces of holomorphic functions and have a wide range of applications. Examples of such spaces include those defined by the coefficients of a Taylor expansion and those defined as the closure of polynomials. This book explains these two Hilbert spaces. Basic examples of Hilbert spaces of holomorphic functions are Hardy spaces, Bergman spaces, and Dirichlet spaces. These spaces have different functional analytical characteristics due to the different ways in which norms are implemented. Historically, the theory of Hardy spaces is the oldest, while the theories of Bergman spaces and Dirichlet spaces are relatively new. Hardy spaces have many applications outside of mathematics, but we will not go into them in this book. Let's take a closer look at the contents of this book. Part I discusses Hilbert spaces of holomorphic functions, which is the main subject of this book, while Part II discusses Banach spaces of holomorphic functions. However, Part II is not a generalization of Part I, but is limited to only the theorems used in Part I. In Chapter 1 of Part I, we state the basic theorems about general Hilbert spaces and bounded linear operators on them, but in a minimal way, in preparation for reading Chapters 2 to 9. However, we do not use the difficult spectral decomposition theorem, and we write it in a self-contained manner. In Chapter 2 of Part I, we investigate Hilbert spaces of holomorphic functions, those defined by sequences and those defined as polynomial closures, and describe their relationship. We also discuss the boundedness of shift operators and point functionals. However, this chapter is also written in a self-contained manner, mainly dealing with material that will help you understand Chapters 3 to 9. The work of Helson and Szegö in 1960 is the problem of drawing the spectrum of a stationary stochastic process whose angle between the past and future is non-zero, as well as the problem of drawing the weights when the analytic projection is bounded on a weighted square integrable Lebesgue space. Both of these are topics on weighted Hardy spaces. Chapter 7 deals with prediction problems of stochastic processes, and Chapter 8 discusses the boundedness of analytical projections.

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Chapter 1　Hilbert space

This chapter describes the basic theorems for Hilbert space and its bounded linear operators, which are used in Chapters 2 and onwards of I. Proofs are given in all cases, except in unavoidable cases, but some use the theorem in Chapter 1 of II. With the exception of Section 8, the subject matter is classic, but Section 8 deals with two recent generalizations of isometric operators. This is used in Chapter 3.

Section 1 Definition of Hilbert space
Section 2 Orthogonal Projection
Section 3 Orthogonal Basis
Section 4 Bounded Linear Functional
Section 5 Bounded Linear Operator
Section 6 Compact Operator
Section 7 Square Root of Positive Linear Operator
Section 8 Operators Close to Isometric Operators

Section 1　Definition of Hilbert space

Definition I-1.1.1. $\mathscr{H}$ is a linear space with complex numbers $\mathbb{C}$ as scalars. For $f, g\in\mathscr{H}$, if $\langle f, g\rangle\in\mathbb{C}$, then $\langle\ ,\ \rangle$ is called an inner product if it satisfies the following axioms.

(1) $\langle f, f\rangle\geq 0 (f\in\mathscr{H})$, and the necessary and sufficient condition for $\langle f, f\rangle=0$ is $f=0$.

(2) $\overline{\langle f, g\rangle}=\langle g, f\rangle (f, g\in\mathscr{H})$, where — denotes the complex conjugate.

(3) $\langle f, g\rangle$ is linear in f and conjugate linear in g.

A linear space with an inner product defined is called an inner product space.

Lemma I-1.1.2. (Schwarz's inequality) Let ⟨ , ⟩ be the inner product on $\mathscr{H}$, and $\|f\|=\sqrt{\langle f, f\rangle}$ $(f\in\mathscr{H})$, then

$$|\langle f, g\rangle|\leq\|f\|\,\|g\| \quad (f, g\in\mathscr{H})$$

holds. Equality holds if and only if there exists $\alpha\in\mathbb{C}$ such that $g=\alpha f$ or $f=0$.

Proof When $g=0$, $\|g\|=0$ and $\langle f, 0\rangle=0$, so both sides are 0 and the inequality holds. Therefore, let $g\neq0$, and let $\alpha=\langle f, g\rangle/\|g\|^2$. Then

$$\begin{aligned}\|f-\alpha g\|^2&=\langle f-\alpha g, f-\alpha g\rangle\\&=\|f\|^2-\alpha\langle g, f\rangle-\overline{\alpha}\langle f, g\rangle+\alpha\overline{\alpha}\|g\|^2\\&=\|f\|^2-2\mathrm{Re}\,\{\alpha\langle g, f\rangle\}+|\alpha|^2\|g\|^2\\&=\|f\|^2-|\langle f, g\rangle|^2/\|g\|^2\end{aligned}$$

This completes the proof. ■

Theorem I-1.1.3. An inner product space ($\mathscr{H}$ with inner product ⟨ , ⟩ is a norm space with the norm defined as $\|f\|=\sqrt{\langle f, f\rangle}$. For the definition of the norm, see Definition II-1.1.1.

Proof It is sufficient to show that $\|\ \|$ is a norm. The axioms (1) and (2) of the norm are obvious. Now we show (3). According to Lemma I-1.1.2, for $f, g\in\mathscr{H}$, we have

$$\begin{aligned}\|f+g\|^2&=\langle f+g, f+g\rangle\\&=\|f\|^2+2\mathrm{Re}\,\langle f, g\rangle+\|g\|^2\\&\leq\|f\|^2+2\|f\|\,\|g\|+\|g\|^2=(\|f\|+\|g\|)^2\end{aligned}$$

This implies (3). This completes the proof. ■

Proposition I-1.1.4. Let $\mathscr{H}$ be an inner product space, and let $f, g\in\mathscr{H}$.

(1) (Midpoint Theorem)

$$\|f+g\|^2+\|f-g\|^2=2\left(\|f\|^2+\|g\|^2\right)$$

(2) (Polarization Identity)

$$\langle f, g\rangle=\frac{1}{4}\left(\|f+g\|^2-\|f-g\|^2+i\|f+ig\|^2-i\|f-ig\|^2\right)$$

Proof Since

$$\|f \pm g\|^2 = \|f\|^2 \pm 2\mathrm{Re}\,\langle f, g\rangle + \|g\|^2,$$

(1) holds. Since

$$\|f+g\|^2 - \|f-g\|^2 = 4\mathrm{Re}\,\langle f, g\rangle,$$
$$\|f+ig\|^2 - \|f-ig\|^2 = 4\mathrm{Re}\,\langle f, ig\rangle = 4\mathrm{Im}\langle f, g\rangle,$$

(2) holds. This completes the proof. ■

Let $\|\ \|$ be a norm, and let $d(f, g) = \|f-g\|$. Then d is a distance, so a normed space is a metric space. For the distance, see Definition II-1.1.2.

Proposition I-1.1.5. In the inner product space, the inner product $\langle f, g\rangle$ is continuous with respect to f and g.

Proof Let $\|f_n - f_0\| \to 0$. Since $|\,\|f_n\| - \|f_0\|\,| \leq \|f_n - f_0\|$, we have $\|f_n\| \to \|f_0\|$, so $\{\|f_n\|\}$ is a bounded sequence. By the Schwarz inequality,

$$\begin{aligned} |\langle f_n, g_n\rangle - \langle f_0, g_0\rangle| &\leq |\langle f_n, g_n - g_0\rangle| + |\langle f_n - f_0, g_0\rangle| \\ &\leq \|f_n\| \cdot \|g_n - g_0\| + \|f_n - f_0\| \cdot \|g_0\| \end{aligned}$$

Thus, if $\|f_n - f_0\| \to 0$ and $\|g_n - g_0\| \to 0$, then $\langle f_n, g_n\rangle \to \langle f_0, g_0\rangle$ holds. ■

If $\lim_{n\to\infty} \|f_n - f\| = 0$, then it is called that f_n converges to f in norm. If $\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle$ for all $g \in \mathcal{H}$, then it is called that f_n weakly converges to f.

Proposition I-1.1.5 indicates that the norm convergent sequence is a weakly convergent sequence.

Definition I-1.1.6. An inner product space $\mathcal{H}$ is called a Hilbert space if it is complete with respect to the norm derived from its inner product.

Example I.-1.1.7. Let X be a compact Hausdorff space. Let (X, Σ, μ) be the probability measure space, Let $L^2(X, \mu)$ be the set of all square-integrable functions with respect to μ on X, where two functions that coincide almost everywhere with respect to μ are identified. For $f, g \in L^2(X, \mu)$, let

$$\langle f, g\rangle = \int_X f\bar{g}d\mu.$$

Then $\langle f, g\rangle$ is an inner product on $L^2(X, \mu)$. $L^2(X, \mu)$ becomes a Hilbert space with respect to this inner product.

Proof By Theorem II-1.1.5, $L^2(X, \mu)$ is complete. Thus it is a Hilbert space. ■

Section 2 Orthogonal Projection

Let $\mathcal{H}$ be a Hilbert space. For $f, g \in \mathcal{H}$, let $\langle f, g\rangle = 0$. Then we say that f and g are orthogonal to each other, and we write $f \perp g$. A subset F, G of $\mathcal{H}$ is said to be orthogonal to each other if $\langle f, g\rangle = 0 \quad (f \in F, g \in G)$ holds, and we write $F \perp G$. Furthermore, the set of all elements that are orthogonal to F is denoted by $F^{\perp}$. $F^{\perp} = \{g \in \mathcal{H}: g \perp F\}$.

Definition I.-1.2.1. Let $\mathcal{H}$ be the inner product space, and let F be its closed subspace. For $h \in \mathcal{H}$, there is an $f \in F$ and a $g \in F^{\perp}$ such that $h = f + g$, Then f is called an orthogonal projection of h on F.

Theorem I.-1.2.2. Let $\mathcal{H}$ be a Hilbert space, and let F be its closed subspace. In this case, any $h \in \mathcal{H}$ can be uniquely decomposed into the form $h = f + g (f \in F, g \in F^{\perp})$ That is, there is a unique orthogonal projection f on F of h.

Proof Uniqueness is clear. We show existence. Let

$$\delta = \mathrm{dist}(h, F) = \inf_{l \in F} \|h - l\|.$$

Then, there exists $f_n \in F$ for $(n = 1, 2, \ldots)$ such that $\|h - f_n\| \to \delta$ as $n \to \infty$. By the midpoint theorem I-1.1.4, since $(f_n + f_m)/2 \in F$, we have

$$\begin{aligned} 4\delta^2 + \|f_n - f_m\|^2 &\leq 4\left\|h - \frac{f_n + f_m}{2}\right\|^2 + \|f_n - f_m\|^2 \\ &= \|(h - f_n) + (h - f_m)\|^2 + \|(h - f_n) - (h - f_m)\|^2 \\ &= 2\|h - f_n\|^2 + 2\|h - f_m\|^2. \end{aligned}$$

If $n\to\infty$ and $m\to\infty$, then $\|f_n-f_m\|\to 0$. Hence f_n is a Cauchy sequence. Since $\mathcal{H}$ is complete, there is an f such that $f=\lim_{n\to\infty} f_n\in\mathcal{H}$. Since F is closed, it follows that $f\in F$ and

$$\delta=\|h-f\|=\min_{l\in F}\|h-l\|.$$

Let $g=h-f$. We show that $g\perp F$. Let t be a real variable. For any $l\in F$,

$$X(t)=\|h-f-tl\|^2.$$

When $t=0$, the minimum value δ^2 is taken. Hence $X'(0)=0$. Since

$$X(t)=\|h-f\|^2-2t\mathrm{Re}\,\langle h-f,\,l\rangle+t^2\|l\|^2,$$

it follows that $X'(0)=-2\mathrm{Re}\,\langle h-f,\,l\rangle=0$. We change l as il. Then $\langle h-f,\,l\rangle=0(l\in F)$. ■

Let $\mathcal{H}$ be a Hilbert space and let F be its closed subspace. Let $G=F^\perp$. Then G is also a closed subspace, and G is called an orthogonal complement of F. By Theorem I-1.2.2, we can show that $G^\perp=(F^\perp)^\perp=F$.

Section 3 Normal Orthogonal Basis

A countable subset $\{e_n\}$ of a Hilbert space $\mathcal{H}$ is called an orthonormal system of $\mathcal{H}$ if it satisfies the condition $\langle e_n, e_m\rangle=\delta_{nm}$, where δ_{nm} is the Kronecker delta. When the smallest closed subspace containing the orthonormal system $\{e_n\}$ denoted as

$$[\{e_n\}]=\{\text{closure of all linear combinations of } \{e_n\}\}$$

is equal to $\mathcal{H}$, then $\{e_n\}$ is called an orthonormal basis of $\mathcal{H}$.

Proposition I-1.3.1. Let $\{e_n\}$ be an orthonormal system of $\mathcal{H}$, and F be a closed subspace of $\mathcal{H}$ generated by $\{e_n\}$. For any $g\in\mathcal{H}$, if

$$f=\sum_{n=1}^{\infty}\langle g, e_n\rangle e_n,$$

then $f\in F$ and f is the orthogonal projection of g onto F.

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