タイトル	Iwanami Studies in Advanced Mathematics　Hilbert Spaces of Analytic Functions (2)　Takahiko Nakazi Professor (Emeritus) of Hokkaido University Mathematical Subject Classification (2000): Primary 46E22; Secondary 47A15, 47B35, 60G10　First published in 2009 in Japanese as Seisoku Kansu no nasu Hiruberuto kukan by Iwanami Shoten, Publishers, Tokyo.
著者	YAMAMOTO, Takanori
引用	北海学園大学学園論集(197): 29-34
発行日	2025-07-25

Iwanami Studies in Advanced Mathematics

Hilbert Spaces of Analytic Functions (2)

Takahiko Nakazi

Professor (Emeritus) of Hokkaido University

Mathematical Subject Classification (2000): Primary 46E22; Secondary 47A15, 47B35, 60G10
First published in 2009 in Japanese as Seisoku Kansu no nasu Hiruberuto kukan by Iwanami Shoten, Publishers, Tokyo.

Translated by Takanori Yamamoto

This paper is a translation of the above book from Chapter 1 Section 3 to Section 5.

Section 3 Normal Orthogonal Basis

A countable subset $\{e_n\}$ of a Hilbert space $\mathscr{H}$ is called an orthonormal system of $\mathscr{H}$ if it satisfies the condition $\langle e_n, e_m\rangle = \delta_{nm}$, where δ_{nm} is the Kronecker delta. When the smallest closed subspace containing the orthonormal system $\{e_n\}$ denoted as

$$[\{e_n\}] = \{\text{closure of all linear combinations of } \{e_n\}\}$$

is equal to $\mathscr{H}$, then $\{e_n\}$ is called an orthonormal basis of $\mathscr{H}$.

Proposition I-1.3.1. Let $\{e_n\}$ be an orthonormal system of $\mathscr{H}$, and F be a closed subspace of $\mathscr{H}$ generated by $\{e_n\}$. For any $g \in \mathscr{H}$, if

$$f = \sum_{n=1}^{\infty} \langle g, e_n\rangle e_n,$$

then $f \in F$ and f is the orthogonal projection of g onto F.

Proof Let $F_N = [e_1, e_2, \cdots, e_N]$. For $g \in \mathscr{H}$, let

$$f_N = \sum_{n=1}^{N} \langle g, e_n\rangle e_n.$$

Then f_N is the orthogonal projection of g onto F_N. If we can show that $\mathrm{dist}(g, F_N) \to \mathrm{dist}(g, F)$ $(N \to \infty)$, then $\|g - f_N\| \to \mathrm{dist}(g, F)$. Since $\|f - f_N\| \to 0 (N \to \infty)$, it follows that $\|g - f\| = \mathrm{dist}(g, F)$. By the proof of Theorem I-1.2.2, f is the orthogonal projection of g onto F.

We show that $\mathrm{dist}(g, F_N) \to \mathrm{dist}(g, F) = \delta$. For any N, $\|g - f_N\| = \mathrm{dist}(g, F_N)$, Since $f_N \in F_N \subset F_{N+1} \subset F$, it follows that

$$\|g - f_N\| \geq \|g - f_{N+1}\| \geq \delta.$$

For any $\varepsilon > 0$, there exists $h \in F$ such that

$$\delta + \varepsilon > \|g - h\| \geq \delta.$$

For this h, there exists N_1 and $h' \in F_{N_1}$ such that $\|h - h'\| < \varepsilon$. Since $\delta + 2\varepsilon > \|g - h'\| \geq \delta$, it follows that

$$\delta + 2\varepsilon > \mathrm{dist}(g, F_{N_1}) = \|g - f_{N_1}\| \geq \delta.$$

Therefore $\lim_{N \to \infty} \mathrm{dist}(g, F_N) = \mathrm{dist}(g, F)$. (end of proof)

Corollary I-1.3.2. (Bessel's inequality) If $\{e_n\}$ is an orthonormal system of $\mathscr{H}$, then

$$\sum_{n=1}^{\infty} |\langle g, e_n \rangle|^2 \leq \|g\|^2 \quad (g \in \mathscr{H}).$$

The equality holds when $\{e_n\}$ is a basis.

Proof We use the same symbols as in Proposition I-1.3.1. Then $\|f - f_N\| \to 0 (N \to \infty)$ and $\|f_N\|^2 = \sum_{n=1}^{N} |\langle g, e_n \rangle|^2$. Since f is the orthogonal projection of g onto F, we have $\|f\| \leq \|g\|$. Since $\|f_N\|^2$ is increasing, we have $\|f_N\|^2 \leq \|g\|^2$. If we set $f = g$, then the latter half follows. (End of proof)

Section 4 Bounded Linear Functionals

If $\mathscr{H}$ is a Hilbert space and $\mathscr{H}^*$ is the set of bounded linear functionals on $\mathscr{H}$, then $\mathscr{H}^*$ is a normed vector space. For $g \in \mathscr{H}$, if

$$\phi_g(f) = \langle f, g \rangle \quad (f \in \mathscr{H})$$

then, by Cauchy-Schwarz inequality, $\phi_g \in \mathscr{H}^*$. In this case, $\|\phi_g\| = \|g\|$. For $\alpha, \beta \in \mathbb{C}$ and $g, h \in \mathscr{H}$,

$$\phi_{\alpha g + \beta h} = \bar{\alpha} \phi_g + \bar{\beta} \phi_h.$$

The following Theorem I-1.4.1 is called the Riesz representation theorem, and shows that $\mathscr{H}$ is identified with $\mathscr{H}^*$.

Theorem I-1.4.1. (Riesz representation theorem) For any bounded linear functional ϕ on $\mathscr{H}$, there exists a unique vector $g \in \mathcal{H}$ such that

$$\phi(f) = \langle f, g \rangle \quad (f \in \mathcal{H}).$$

Proof Let Ker $\phi = \{f \in \mathcal{H}: \phi(f) = 0\}$. Since ϕ is bounded, Ker ϕ is a closed subspace of $\mathcal{H}$. If Ker $\phi = \mathcal{H}$, then $\phi(f) = \langle f, 0 \rangle (f \in \mathcal{H})$, so the theorem is true for this case. If Ker $\phi \neq \mathcal{H}$, then there exists h of norm 1 such that $h \perp$ Ker ϕ. Since $h \notin$ Ker ϕ, $\phi(h) \neq 0$. For $f \in \mathcal{H}$, $f - (\phi(f)/\phi(h))h \in$ Ker ϕ. Therefore, for $f \in \mathcal{H}$,

$$\begin{aligned}\phi(f) = \phi(f)\langle h, h \rangle &= \left\langle \frac{\phi(f)}{\phi(h)} h, \overline{\phi(h)} h \right\rangle \\ &= \left\langle f, \overline{\phi(h)} h \right\rangle + \left\langle \frac{\phi(f)}{\phi(h)} h - f, \overline{\phi(h)} h \right\rangle \\ &= \left\langle f, \overline{\phi(h)} h \right\rangle.\end{aligned}$$

Therefore, if $g = \overline{\phi(h)} h$, then $\phi(f) = \langle f, g \rangle$.

If $\langle f, g_1 \rangle = \langle f, g_2 \rangle$ for all $f \in \mathcal{H}$, then in particular, for $f = g_1 - g_2$, $\langle g_1 - g_2, g_1 - g_2 \rangle = 0$, so $g_1 = g_2$, which proves uniqueness. (End of proof)

Theorem I-1.4.2. The unit ball of $\mathcal{H}$ is weakly compact.

Proof By Theorem I-1.4.1 and the remarks, $\mathcal{H}^* = \mathcal{H}$, so this is the result of Theorem II-1.1.9. (End of proof)

Corollary I-1.4.3. Let M be a closed subspace of $\mathcal{H}$ and $\phi \in \mathcal{H}^*$. If ϕ is not zero on M, then there is a unique solution $F = F_M \in M$ to the extremal problem

$$\sup \{\mathrm{Re}\, \phi(f): f \in M, \|f\| \leq 1\}$$

Proof If

$$\alpha = \sup \{\mathrm{Re}\, \phi(f): f \in M, \|f\| \leq 1\}$$

then there exists $\{f_n\} \subset M$ such that $\|f_n\| \leq 1$ and Re $\phi(f_n) \to \alpha$. By Theorem I-1.4.2, there exists a subsequence $\{f_{n_j}\}$ of $\{f_n\}$, and f_{n_j} converges to $F \in \mathcal{H}$, which is a weak topology. By Theorem II-1.1.11, M is closed in the weak topology of $\mathcal{H}$, so $F \in M$ and Re $\phi(F) = \alpha$.

We shall show that F is a unique solution. If G is another solution to this extremal problem, then $tF+(1-t)G$ also solves the same extremal problem for any $0<t<1$.

In this case, since ϕ is not zero, for any t

$$\|tF+(1-t)G\|=1=\|tF\|+\|(1-t)G\|$$

Therefore, $\langle tF,(1-t)G\rangle=\|tF\|\cdot\|(1-t)G\|$.

By Lemma I-1.1.2, $(1-t)G$ indicates a positive scalar multiple of tF, so $G=F$. (End of proof)

Theorem I-1.4.4. If $\mathcal{H}$ is separable, then the unit ball of $\mathcal{H}$ endowed with the weak topology is metrizable.

Proof Let $\mathcal{H}_1$ be the unit ball of $\mathcal{H}$, and $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$. For $f,g\in\mathcal{H}_1$, if

$$d(f,g)=\sum_{j=1}^{\infty}2^{-(j+1)}|\langle f-g,e_j\rangle|$$

then d is the distance of $\mathcal{H}_1$. If $f_n\in\mathcal{H}_1$ weakly converges to $f\in\mathcal{H}_1$, then $d(f_n,f)\to 0$, since $\langle f_n-f,e_j\rangle\to 0$ and $|\langle f_n-f,e_j\rangle|\leq 2$, $d(f_n,f)\to 0$. Conversely, if $d(f_n,f)\to 0$, then $\langle f_n-f,e_j\rangle\to 0$. However, for any $x\in\mathcal{H}_1$, we can write $x=\sum_{j=1}^{\infty}\alpha_j e_j$ by Proposition I-1.3.1. Therefore, using $\|f_n-f\|\leq 2$ and the Cauchy-Schwarz inequality (I-1.1.2),

$$|\langle f_n-f,x\rangle|\leq 2\left\|x-\sum_{j=0}^{\ell}\alpha_j e_j\right\|+\left|\left\langle f_n-f,\sum_{j=0}^{\ell}\alpha_j e_j\right\rangle\right|.$$

This implies that $\langle f_n-f,x\rangle\to 0$. (End of proof)

Section 5 Bounded Linear Operator

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and T be a linear operator from $\mathcal{H}$ to $\mathcal{K}$.

T is said to be bounded if there exists a positive constant γ such that

$$\|Tf\|_{\mathcal{K}}\leq\gamma\|f\|_{\mathcal{H}}\quad(f\in\mathcal{H}).$$

$\|T\|$ represents the lower bound of such γ.

The set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ is written by $\mathcal{B}(\mathcal{H},\mathcal{K})$. If $\mathcal{H}=\mathcal{K}$, then $\mathcal{B}(\mathcal{H})$ in short. It is well known that T is bounded if and only if T is continuous.

The set of all $\lambda\in\mathbb{C}$ for which $T-\lambda I$ is not invertible for $T\in\mathcal{B}(\mathcal{H})$ is denoted by $\sigma(T)$, where I is the identity operator on $\mathcal{H}$. $\sigma(T)$ is called the spectrum of T, and is known to be a compact set of $\mathbb{C}$ that is not the empty set.

Lemma I-1.5.1. For any $g\in\mathcal{K}$, if there exists a bounded linear functional Φ_g on $\mathcal{H}$ such that

$$|\Phi_g(f)| \leq \gamma \|f\|_{\mathcal{H}} \|g\|_{\mathcal{K}} \quad (f \in \mathcal{H})$$

and there exists a positive constant γ such that

$$|\Phi_g(f)| \leq \gamma \|f\|_{\mathcal{H}} \|g\|_{\mathcal{K}} \quad (f \in \mathcal{H})$$

then there exists a bounded linear operator S from $\mathcal{K}$ to $\mathcal{H}$ such that

$$\Phi_g(f) = \langle f, Sg \rangle_{\mathcal{H}} \quad (g \in \mathcal{K})$$

Proof By the Riesz representation theorem I-1.4.1, there exists a unique $h \in \mathcal{H}$ such that $\Phi_g(f) = \langle f, h \rangle_{\mathcal{H}}$, so let $Sg = h$. In this case, it is clear that $\|S\| \leq \gamma$. (End of proof)

Proposition I-1.5.2. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, there exists a unique $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$\langle Tf, g \rangle_{\mathcal{K}} = \langle f, Sg \rangle_{\mathcal{H}} \quad (f \in \mathcal{H}, g \in \mathcal{K}).$$

In this case, we write $S = T^*$ and call it the conjugate operator of T.

Proof For $g \in \mathcal{K}$,

$$\Phi_g(f) = \langle Tf, g \rangle_{\mathcal{K}} \quad (f \in \mathcal{H})$$

By the Cauchy-Schwarz inequality, Φ_g satisfies Lemma I-1.5.1, so there exists a bounded linear operator S from $\mathcal{K}$ to $\mathcal{H}$ such that

$$\langle Tf, g \rangle_{\mathcal{K}} = \Phi_g(f) = \langle f, Sg \rangle_{\mathcal{H}} \quad (f \in \mathcal{H}, g \in \mathcal{K}).$$

We shall show the uniqueness. If there exists another $S' \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\langle Tf, g \rangle_{\mathcal{K}} = \langle f, S'g \rangle_{\mathcal{H}}$ $(f \in \mathcal{H}, g \in \mathcal{K})$, then $\langle f, Sg - S'g \rangle_{\mathcal{H}} = 0 (f \in \mathcal{H})$. Therefore, $(S - S')g = 0 (g \in \mathcal{K})$, so $S = S'$. (End of proof)

Corollary I-1.5.3. If $T \in \mathcal{B}(\mathcal{H})$, then $\mathcal{H} = \mathrm{Ker}\, T^* \oplus [\mathrm{Ran} T]$, where $\mathrm{Ker}\, T^* = \{f \in \mathcal{H} : T^* f = 0\}$ and $\mathrm{Ran} T = T\mathcal{H}$.

Proof This is clear from the above proposition.

Theorem I-1.5.4. (Schur's theorem) Let $T = (a_{ij})$ be the representation matrix of the linear operator T with respect to an orthonormal basis on ℓ^2. If there exist $0 < \gamma < \infty$ and

$0 < h_j < \infty (1 \leq j < \infty)$ such that

$$\sum_{j=1}^{\infty} |a_{ij}| h_j \leq \gamma h_i \quad (i \geq 1),$$

$$\sum_{i=1}^{\infty} |a_{ij}| h_i \leq \gamma h_j \quad (j \geq 1),$$

then T is bounded on ℓ^2 and $\|T\| \leq \gamma$.

Proof Let $(f_j) \in \ell^2$ and $1 \leq n < \infty$. Let fix $1 \leq i < \infty$. Then

$$\begin{aligned}\left|\sum_{j=1}^{n} a_{ij} f_j\right| &\leq \sum_{j=1}^{n} |a_{ij}| h_j^{1/2} h_j^{-1/2} |f_j| \\ &\leq \left(\sum_{j=1}^{n} |a_{ij}| h_j\right)^{1/2} \left(\sum_{j=1}^{n} |a_{ij}| h_j^{-1} |f_j|^2\right)^{1/2} \\ &\leq \sqrt{\gamma}\, h_i^{1/2} \left(\sum_{j=1}^{n} |a_{ij}| h_j^{-1} |f_j|^2\right)^{1/2}.\end{aligned}$$

Therefore, from the assumption,

$$\begin{aligned}\sum_{i=1}^{n} \left|\sum_{j=1}^{n} a_{ij} f_j\right|^2 &\leq \gamma \sum_{i=1}^{n} \left(h_i \sum_{j=1}^{n} |a_{ij}| h_j^{-1} |f_j|^2\right) \\ &= \gamma \sum_{j=1}^{n} h_j^{-1} |f_j|^2 \left(\sum_{i=1}^{n} |a_{ij}| h_i\right) \\ &\leq \gamma \sum_{j=1}^{n} h_j^{-1} |f_j|^2 \times \gamma h_j = \gamma^2 \sum_{j=1}^{n} |f_j|^2.\end{aligned}$$

(End of Proof)