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Iwanami Studies in Advanced Mathematics

Hilbert Spaces of Analytic Functions (3)

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This paper is a translation of the above book from Chapter 1 Section 6 to Section 7.

Section 6 Compact Operators

Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces. The unit ball of $\mathcal{H}$ is denoted by $\mathcal{H}_1$. Then $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$ is a compact operator if and only if the closure of $T\mathcal{H}_1$ is compact in the norm topology of $\mathcal{K}$.

Theorem I.-1.6.1. Let $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$. Then the necessary and sufficient condition for T to be compact is that f_n weakly converges to 0 in $\mathcal{H}$, i.e., for any $h\in\mathcal{H}$, $\lim_{n\to\infty}\langle f_n, h\rangle_{\mathcal{H}}=0$, then $\lim_{n\to\infty}\|Tf_n\|_{\mathcal{K}}=0$.

Proof. We shall show that if T is compact and f_n weakly converges to 0 in $\mathcal{H}$, then $\|Tf_n\|_{\mathcal{K}}\to 0(n\to\infty)$. If not, select a subsequence if necessary, then there exists an ε such that $\|Tf_n\|_{\mathcal{K}}\geq\varepsilon>0$ $(n=1, 2, \cdots)$. By the Banach-Steinhaus theorem II-1.3.1, if f_n weakly converges to 0, then $\sup_n\|f_n\|_{\mathcal{H}}<\infty$. Since T is compact, $\{Tf_n\}$ is a subset of some norm compact set in $\mathcal{K}$. Thus, there is a subsequence $\{f_{nj}\}$ in $\mathcal{H}$ such that $\{Tf_{nj}\}$ converges to $g\in\mathcal{K}$ in norm. In particular, $\{Tf_{nj}\}$ weakly converges to g. Therefore $g=0$. It follows that $\|Tf_{nj}\|_{\mathcal{K}}\to 0$. This is a contradiction.

Conversely, suppose T is not compact. Then the closure of $T\mathcal{H}_1$ in $\mathcal{K}$ is not norm-compact. Thus, there is a $\{f_n\}\subset\mathcal{H}_1$ such that $\{f_n\}$ does not normally converge with any subsequence of $\{Tf_n\}$. By the Alaoglu's theorem II.-1.1.9, $\mathcal{H}_1$ is compact in weak topology, so there is a weakly converging subsequence $\{f_{nj}\}$. If $F_j=f_{nj}(j=1, 2, \cdots)$, then $\{F_j\}$ converges weakly, but TF_j does not converge in norm. (End of proof)

T is a finite rank operator if $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and $T\mathcal{H}$ is a finite dimensional subspace of $\mathcal{K}$.

Proposition I-1.6.2.

(1) If T is a finite rank operator on $\mathcal{H}$, then there exists $0\leq n<\infty$ such that

$$Tf=\sum_{j=0}^{n}\langle f, G_j\rangle_{\mathcal{H}}F_j \ (f\in\mathcal{H})$$

where $\{G_j\}_{j=1}^{n}\subset\mathcal{H}$ and $\{F_j\}_{j=1}^{n}\subset\mathcal{K}$.

(2) If T is a finite rank operator, then T is a compact operator.

Proof (1) Since $T\mathcal{H}$ is a finite-dimensional subspace of $\mathcal{K}$, if $\{F_j\}_{j=1}^{n}$ is its orthonormal basis, then for any $f\in\mathcal{H}$,

$$Tf=\sum_{j=1}^{n}\alpha_j(f)F_j,\ \alpha_j(f)\in\mathbb{C}\quad(1\leq j\leq n).$$

For any j, $\langle Tf, F_j\rangle_{\mathcal{K}}=\alpha_j(f)$, but since T is bounded, α_j is a bounded linear functional on $\mathcal{H}$. By Riesz's representation theorem (I-1.4.1), there exists $G_j\in\mathcal{H}$ such that $\alpha_j(f)=\langle f, G_j\rangle_{\mathcal{H}}$.

(2) By (1), if f_ℓ converges weakly to 0, then

$$\|Tf_\ell\|_{\mathcal{K}}^2=\sum_{j=1}^{n}\left|\langle f_\ell, G_j\rangle_{\mathcal{H}}\right|^2$$

Therefore, $\|Tf_\ell\|_{\mathcal{K}}\to 0(\ell\to\infty)$. By Theorem I-1.6.1, T is compact. (End of proof)

Remark I-1.6.3. If T is compact, then it is known that T can be approximated by some finite rank operator in operator norm. By Theorem II-1.3.3 and Proposition I-1.6.2 (2), the converse is also true.

Section 7 Products and Square Roots of Positive Linear Operators

$T\in\mathcal{B}(\mathcal{H})$ is said to be positive when

$$\langle Tf, f\rangle\geq 0\quad(f\in\mathcal{H})$$

holds, and we write $T\geq 0$. If $T, S\in\mathcal{B}(\mathcal{H})$ and $T-S\geq 0$, then we write $T\geq S$. Even if $T, S\in\mathcal{B}(\mathcal{H}), S\geq 0$ and $T\geq 0$, this does not necessarily mean that $TS\geq 0$. In this section, we will show that if $TS=ST$, then $TS\geq 0$. We will also use this to show that there is a unique square root of T.

Lemma I-1.7.1. If $T, S\in\mathcal{B}(\mathcal{H})$, $T\geq 0, S\geq 0$ and $TS=ST$, then $TS^2\geq 0$.

Proof Since $TS=ST$ and $T\geq 0$, it follows that

$$\langle TS^2 f, f\rangle=\langle TSf, Sf\rangle\geq 0$$

Therefore, $TS^2\geq 0$. (End of proof)

Lemma I-1.7.2. If $S\in\mathcal{B}(\mathcal{H})$ and $S\geq 0$, then for each k there exists $S_k\geq 0$ which is a polynomial of S and can be written as $S=\sum_{k=1}^{\infty} S_k^2$. Here the convergence is strong.

Proof Without loss of generality, we suppose that $0\leq S\leq I$. If we define $\{S_k\}_{k=1}^{\infty}$ as $S_1=S$ and $S_{k+1}=S_k(I-S_k)$, then S_k is a polynomial of S, and we have $0\leq S_k\leq I (1\leq k\leq\infty)$. We shall show this by induction. This holds when $k=1$. If $0\leq S_k\leq I$ holds for some k, then by Lemma I-1.7.1, $S_k^2(I-S_k)\geq 0$ and $S_k(I-S_k)^2\geq 0$, therefore

$$S_{k+1}=S_k(I-S_k)=S_k^2(I-S_k)+S_k(I-S_k)^2\geq 0$$

and

$$I-S_{k+1}=I-S_k(I-S_k)=(I-S_k)+S_k^2\geq 0$$

and therefore $0\leq S_{k+1}\leq I$. Then

$$\sum_{k=1}^{n} S_k^2=\sum_{k=1}^{n}(S_k-S_{k+1})=S-S_{n+1}\leq S.$$

Hence,

$$\sum_{k=1}^{\infty}\|S_k f\|^2=\sum_{k=1}^{\infty}\langle S_k f, S_k f\rangle\leq\langle Sf, f\rangle.$$

Therefore,

$$\lim_{n\to\infty}\left\|\sum_{k=1}^{n} S_k^2 f-Sf\right\|=\lim_{n\to\infty}\|S_{n+1}f\|=0$$

and so $\sum_{k=1}^{\infty} S_k^2 f=Sf (f\in\mathcal{H})$ holds. (End of proof)

Theorem I-1.7.3. If $T, S\in\mathcal{B}(\mathcal{H})$, $T\geq 0, S\geq 0$ and $TS=ST$, then $TS\geq 0$.

Proof By Lemma I-1.7.1 and I-1.7.2. (End of proof)

Lemma I-1.7.4. If $T\in\mathcal{B}(\mathcal{H})$ and $T\geq 0$, then

$$|\langle Tf, g\rangle|^2 \leq \langle Tf, f\rangle\langle Tg, g\rangle \quad (f, g\in\mathcal{H}).$$

Proof If $[f, g]=\langle Tf, g\rangle$, then $[f, g]$ satisfies (2) and (3) of the definition of the inner product I-1.1.1. By the proof of Schwarz's inequality I-1.1.2, it follows that $|[f, g]|^2\leq[f, f][g, g]$. (End of proof)

Lemma I-1.7.5. If $T_n\in\mathcal{B}(\mathcal{H})$ and $0\leq T_n\leq T_{n+1}\leq I$, then there exists a positive $T\in\mathcal{B}(\mathcal{H})$ such that for any $f\in\mathcal{H}$, $\|T_nf-Tf\|\to 0(n\to\infty)$.

Proof We will show that $\|T_nf-T_mf\|\to 0(n, m\to\infty)$. Since $\mathcal{H}$ is complete, this implies that there exists $g_f\in\mathcal{H}$ such that $\lim_{n\to\infty} T_nf=g_f$. Let $Tf=g_f$. Then $T\in\mathcal{B}(\mathcal{H})$, and the proof will be finished. By Lemma I-1.7.4, if $n>m$, then

$$\begin{aligned}
&\|T_nf-T_mf\|^4\\
&=\langle(T_n-T_m)f, (T_n-T_m)f\rangle^2\\
&\leq\langle(T_n-T_m)f, f\rangle\langle(T_n-T_m)(T_n-T_m)f, (T_n-T_m)f\rangle\\
&\leq\langle(T_n-T_m)f, f\rangle\langle(T_n-T_m)f, (T_n-T_m)f\rangle\\
&=\langle(T_n-T_m)f, f\rangle\|(T_n-T_m)f\|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|T_nf-T_mf\|^2&\leq\langle(T_n-T_m)f, f\rangle\\
&=\langle T_nf, f\rangle-\langle T_mf, f\rangle.
\end{aligned}$$

Since $0\leq\langle T_nf, f\rangle\leq\|f\|^2$, it follows that there exists $\lim_{n\to\infty}\langle T_nf, f\rangle$. Therfore $\|T_nf-T_mf\|\to 0(n, m\to\infty)$. (End of proof)

Theorem I-1.7.6. Let $T\in\mathcal{B}(\mathcal{H})$ and $T\geq 0$. Then there exists a unique positive $S\in\mathcal{B}(\mathcal{H})$ such that $S^2=T$. Then $ST=TS$. We write $S=T^{1/2}$.

Proof Without loss of generality, we suppose that $0\leq T\leq I$. We shall prove the existence of S. If we define $\{S_k\}_{k=1}^{\infty}$ as

$$S_0=0 \text{ and } S_{k+1}=S_k+\frac{1}{2}(T-S_k^2),$$

then $S_kT=TS_k$ and

$$0=S_0\leq S_1\leq\cdots\leq I$$

holds. In fact,

$$I-S_{k+1}=\frac{1}{2}(I-S_k)^2+\frac{1}{2}(I-T)$$

Since $I\geq S_{k+1}$ and

$$S_{k+1}-S_k=\frac{1}{2}\{(I-S_{k-1})+(I-S_k)\}(S_k-S_{k-1}),$$

by induction, we have $S_{k+1}\geq S_k$. Here we use Lemma I-1.7.1. By Lemma I-1.7.5, there exists $S\in\mathcal{B}(\mathcal{H})$ such that $\lim_{k\to\infty}\|S_kf-Sf\|=0(f\in\mathcal{H})$. Then, since $T=S^2$ and $S_kT=TS_k(0\leq k<\infty)$, we have $ST=TS$.

We shall prove the uniqueness of S. Let $0\leq K\leq I$ and $K^2=T$. Since $KT=KKK=TK$ and S is the limit of a polynomial in T, we have $KS=SK$. Hence, $(S+K)(S-K)=S^2-K^2=0$. From the first half of the proof, there exist positive operators $R_S, R_K\in\mathcal{B}(\mathcal{H})$ such that $R_S^2=S$ and $R_K^2=K$. Let $y=(S-K)x$ for $x\in\mathcal{H}$, then

$$\begin{aligned}&\|R_Sy\|^2+\|R_Ky\|^2\\&=\langle(S+K)y, y\rangle\\&=\langle(S+K)(S-K)x, y\rangle=0.\end{aligned}$$

This implies that $R_Sy=R_Ky=0$. Hence $Sy=R_S^2y=0$ and $Ky=R_K^2y=0$. Therefore

$$\begin{aligned}\|(S-K)x\|^2&=\langle(S-K)(S-K)x, x\rangle\\&=\langle(S-K)y, x\rangle=0\end{aligned}$$

This implies that $S=K$. (End of proof)

Corollary I-1.7.7. Let $T\in\mathcal{B}(\mathcal{H})$, $T\geq 0$ and m be a positive constant. Then $\|T^{-1}\|\leq m$ if and only if for any $g\in\mathcal{H}$ and $f=Tg$, $m\|f\|^2\geq|\langle f, g\rangle|$.

Proof If $m\|f\|^2\geq|\langle f, g\rangle|$, then $m\|Tg\|^2\geq|\langle Tg, g\rangle|$. Let $S=T^{1/2}$. Then $m\langle TSg, Sg\rangle\geq\|Sg\|^2$ and $S\mathcal{H}$ is dense in $\mathcal{H}$. Hence $m\langle Th, h\rangle\geq\|h\|^2(h\in\mathcal{H})$. This implies that $\|S^{-1}\|\leq\sqrt{m}$. Therefore $\|T^{-1}\|\leq m$. The converse is obvious. (End of proof)