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Hilbert Spaces of Analytic Functions (4)
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This paper is a translation of the above book from Chapter 1 Section 8 to Chapter 2 Section 1.

Section 8 Operators Close to Isometric Operators

Let $\mathcal{H}$ be a Hilbert space. If $T\in\mathcal{B}(\mathcal{H})$ satisfies $\|Tx\|=\|x\|$ $(x\in\mathcal{H})$, that is, if $T^*T=I$, then T is called an isometric operator. If $T^*T=TT^*=I$, then T is called a unitary operator.

Here, we define two generalizations of the isometric operator. Let $T\in\mathcal{B}(\mathcal{H})$. If

$$\|T^2x\|^2+\|x\|^2\leq 2\|Tx\|^2 \quad (x\in\mathcal{H}),$$

then it is said that T satisfies condition (A). If

$$\|Tx+y\|^2\leq 2\left(\|x\|^2+\|Ty\|^2\right) \quad (x,y\in\mathcal{H}),$$

then it is said that T satisfies condition (B).

The isometric operator clearly satisfies condition (A) and condition (B). Proposition I-1.8.2 shows the relationship between condition (A) and condition (B).

Lemma I-1.8.1. If T satisfies condition (B), then T^*T is invertible.

Proof In condition (B), if $x=0$ then $\|Ty\|^2\geq\frac{1}{2}\|y\|^2(y\in\mathcal{H})$, so T^*T is one-to-one on $\mathcal{H}$ and $T^*T\mathcal{H}$ is a closed set. In fact, one-to-one is obvious. If $T^*Ty_n\to z$, then $\|Ty_n\|^2\geq\frac{1}{2}\|y_n\|^2$ which implies that $\{y_n\}$ is the Cauchy sequence. Since y_n converges to a $y\in\mathcal{H}$, so $T^*Ty=z$, and so $T^*T\mathcal{H}$ is a closed set. By Corollary I.-1.5.3, T^*T is invertible. (End of proof)

If T satisfies condition (B), then $T(T^*T)^{-1/2}$ is an isometric operator, while Proposition I.-1.8.2 indicates that $T(T^*T)^{-1}$ satisfies condition (A).

Proposition I-1.8.2. If $T\in\mathcal{B}(\mathcal{H})$ satisfies condition (B), then $T(T^*T)^{-1}$ satisfies condition (A).

Proof Let $y=(T^*T)^{-1/2}z$. Since $T(T^*T)^{-1/2}$ is an isometric operator, it follows from condition (B) that

$$\left\|Tx+(T^*T)^{-1/2}z\right\|^2\leq 2\left(\|x\|^2+\|z\|^2\right).$$

If we define $L\colon \mathcal{H}\oplus\mathcal{H}\to\mathcal{H}$ as

$$L(x, z)=Tx+(T^*T)^{-\frac{1}{2}}z,$$

then $\|L\|\leq\sqrt{2}$. Therefore $LL^*\leq 2I$, where $L^*\colon \mathcal{H}\to\mathcal{H}\oplus\mathcal{H}$ is defined by

$$L^*=\left(T^*,(T^*T)^{-1/2}\right).$$

Here, the norm of $\mathcal{H}\oplus\mathcal{H}$ is defined by $\|(x, y)\|=\left\{\|x\|^2+\|y\|^2\right\}^{1/2}$.

Since $LL^*\leq 2I$, it follows that

$$TT^*+(T^*T)^{-1}\leq 2I.$$

If $S=T(T^*T)^{-1}$, then $T=S(S^*S)^{-1}$, so

$$S(S^*S)^{-1}(S^*S)^{-1}S^*+S^*S\leq 2I.$$

Therefore,

$$I+(S^*)^2S^2\leq 2S^*S.$$

Therefore, $T(T^*T)^{-1}$ satisfies condition (A). (End of proof)

Proposition I.-1.8.3. Let T be an isometric operator on $\mathcal{H}$. If $\mathcal{H}_\infty=\bigcap_{n=0}^{\infty}T^n\mathcal{H}$, then $T\mathcal{H}_\infty=\mathcal{H}_\infty$ and $T^*\mathcal{H}_\infty=\mathcal{H}_\infty$. The restriction of T to $\mathcal{H}_\infty$ becomes a unitary operator.

Proof Since $T^*T=I$, it follows that $T^*\mathcal{H}_\infty=\mathcal{H}_\infty$, and $T\mathcal{H}_\infty=\mathcal{H}_\infty$. Let T_∞ denote the restriction of T to $\mathcal{H}_\infty$. Then T_∞ becomes an isometric operator on $\mathcal{H}_\infty$. For any $x\in\mathcal{H}_\infty$, there exists a $y\in\mathcal{H}_\infty$ satisfying $T_\infty y=x$. Therefore

$$T_\infty T_\infty^* x = T_\infty T_\infty^* T_\infty y = T_\infty y = x.$$

Thus, T_∞ is a unitary operator. (End of proof)

Proposition I-1.8.3 is also a corollary of the following Theorem I-1.8.4.

Theorem I-1.8.4. Suppose T satisfies condition (A). If $\mathcal{H}_\infty = \bigcap_{n=0}^\infty T^n\mathcal{H}$, then $T\mathcal{H}_\infty = \mathcal{H}_\infty$ and $T^*\mathcal{H}_\infty = \mathcal{H}_\infty$, and the restriction of T to $\mathcal{H}_\infty$ is a unitary operator.

Proof If T satisfies condition (A), then

$$\|T^2x\|^2 + \|x\|^2 \leq 2\|Tx\|^2 \quad (x \in \mathcal{H}).$$

Therefore if $x \neq 0$, then $Tx \neq 0$. Hence T is one-to-one on $\mathcal{H}$.

(Claim 1) $\|Tx\| \geq \|x\| (x \in \mathcal{H})$. By condition (A), for $n = 0, 1, 2, \cdots$,

$$\|T^{n+2}x\|^2 - \|T^{n+1}x\|^2 \leq \|T^{n+1}x\|^2 - \|T^n x\|^2.$$

This implies that the sequence $\{\|T^n x\|\}$ is non-decreasing. Therefore (Claim 1) holds. In the following, we shall show that $\{\|T^n x\|\}$ is non-decreasing. For some $k \geq 0$ and some $x \in \mathcal{H}$, if $\|T^{k+1}x\| < \|T^k x\|$, then from the above inequality, for any $n \geq k$, $\|T^{n+1}x\| < \|T^n x\|$ holds. Since $\{\|T^n x\|\}_{n \geq k}$ is decreasing, there exists $\lim_{n\to\infty} \|T^n x\|$, and therefore

$$0 = \lim_{n\to\infty} \left(\|T^{n+1}x\|^2 - \|T^n x\|^2\right)$$
$$\leq \|T^{k+1}x\|^2 - \|T^k x\|^2 < 0$$

This is a contradiction. Hence, the sequence $\{\|T^n x\|\}$ is non-decreasing.

(Claim 2) T is an isometric operator on $\mathcal{H}_\infty$. Let $x \in \mathcal{H}_\infty$. Let $y_1 = x$. For $n \geq 2$, $y_n \in \mathcal{H}_\infty$ is defined by $Ty_n = y_{n-1}$, since T is one-to-one. Hence $T^{n-1}(Ty_n) = T^n y_n = T^{n-1} y_{n-1}$. Since T satisfies condition (A), this implies that for $n = 0, 1, 2, \cdots$,

$$\|y_{n+2}\|^2 + \|y_n\|^2 \leq 2\|y_{n+1}\|^2.$$

On the other hand, by (Claim 1), $\|y_n\|^2 = \|Ty_{n+1}\|^2 \geq \|y_{n+1}\|^2$, so $\{\|y_n\|\}$ is decreasing. From the above inequality, $0 \leq \|y_n\|^2 - \|y_{n+1}\|^2 \leq \|y_{n+1}\|^2 - \|y_{n+2}\|^2$, and therefore for any $k \geq 0$,

$$0 \leq \|y_n\|^2 - \|y_{n+1}\|^2 \leq \|y_{n+k}\|^2 - \|y_{n+k+1}\|^2$$

holds. Since $\lim_{k\to\infty} \|y_{n+k}\|$ exists, $\|y_n\| = \|y_{n+1}\|$. Therefore, $\|Ty_{n+1}\| = \|y_{n+1}\|$ $(n = 0, 1, 2, \cdots)$ holds. Since $\|Ty_1\| = \|y_1\|$, $\|Tx\| = \|x\|$.

(Claim 3) $T^*\mathcal{H}_\infty = \mathcal{H}_\infty$. Since T is an isometric operator on $\mathcal{H}_\infty$, we have $\langle (T^*T - I)x, x\rangle = 0$ ($x \in \mathcal{H}_\infty$). On the other hand, from (Claim 1), $T^*T - I \geq 0$ on $\mathcal{H}$, so $T^*Tx = x$ ($x \in \mathcal{H}_\infty$), and therefore $T^*\mathcal{H}_\infty = \mathcal{H}_\infty$.

It follows from (Claim 2) and (Claim 3) that the restriction of T to $\mathcal{H}_\infty$ is a unitary operator, as in the proof of Proposition I-1.8.3. (End of proof)

Note I-1.8.5. From Proposition I-1.8.3, the von Neumann-Wold decomposition is easily derived [52]. Theorem I-1.8.4 is due to Shimorin [71]. When condition (A) holds under equality, it is called a 2-isometry, Agler [2].

Chapter 2　Spaces of Analytic Functions

This chapter describes Hilbert spaces consisting of analytic functions on the open unit disc of the complex plane. Section 2 determines a reproducing kernel Hilbert space which is defined by the coefficients of a Taylor expansion. We also describe the reproducing kernel. Section 3 determines a reproducing kernel Hilbert space which is defined as the closure of polynomials. We also investigate the reproducing kernel. Section 4 describes the relationship between Hardy space, Bergman space and Hilbert space. We also prove Douglas's theorem, which expresses the Dirichlet integral in terms of a local Dirichlet integral. This chapter does not use the concepts discussed in Section II.

Section 1　Hilbert Space

Section 2　Sequence and Analytic Function

Section 3　Polynomial and Analytic Function

Section 4　Relations between Hardy space, Bergman space, and Dirichlet Space

Section 1　Hilbert Space

Let D be the unit disc of the complex plane $\mathbb{C}$, i.e., $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{P}$ be the set of polynomials in z. Let $H(D)$ denote the set of analytic functions on D. Let $\mathcal{H}$ denote the subspace of $H(D)$. Their inner product is written by $\langle\ ,\ \rangle$, and their norm by $\|\cdot\|$. For each $a \in D$,

$$\tau_a(f)=f(a) \quad (f\in\mathcal{H})$$

and this is called the point functional. In this case, we assume that τ_a is always bounded on $\mathcal{H}$. Furthermore, two linear operators T_z and S_z on $\mathcal{H}$, which are important for studying the Hilbert space $\mathcal{H}$, are defined as follows:

$$T_zf=zf,\, S_zf=\frac{f-f(0)}{z} \quad (f\in\mathcal{H}).$$

If $T_zf\in\mathcal{H}$ for $f\in\mathcal{H}$, then $S_zT_zf=f$, and if $S_zf\in\mathcal{H}$, then $(T_zS_z+\tau_0)f=f$. Therefore,

$$S_zT_z-T_zS_z=\tau_0$$

For $\phi\in H(D)$, let

$$(T_\phi f)(z)=\phi(z)f(z) \quad (f\in\mathcal{H}).$$

If $\phi\in H(D)$ and $T_\phi\mathcal{H}\subset\mathcal{H}$, then ϕ is called a multiplier on $\mathcal{H}$. T_z is called the forward shift operator, and S_z is called the backward shift operator. T_z is studied in detail in Chapter 3, and S_z is studied in detail in Chapter 4.

Since τ_a is bounded on $\mathcal{H}$ for $a\in D$, and $\mathcal{H}$ is a Hilbert space, by Riesz's representation theorem (I-1.3.1), there exists $K_a\in\mathcal{H}$ such that $\|\tau_a\|=\|K_a\|$ and

$$\tau_a(f)=\langle f, K_a\rangle \quad (f\in\mathcal{H}).$$

K_a is called the reproducing kernel for a. If $T_zf\in\mathcal{H}$ holds for all $f\in\mathcal{H}$, then $\mathcal{H}$ is said to be an invariant subspace of T_z. The same is true for S_z.

If $\mathcal{H}$ is an invariant subspace of T_z, then for all $a\in D$,

$$T_z^*K_a=\bar{a}K_a.$$

In fact, by the definition of T_z^*,

$$\langle f, T_z^*K_a\rangle=\langle zf, K_a\rangle=af(a)=\langle f, \bar{a}K_a\rangle$$

holds for any $f\in\mathcal{H}$, so $T_z^*K_a=\bar{a}K_a$. For $a\in D$,

$$S_{z,a}f=\frac{f-f(a)}{z-a} \quad (f\in\mathcal{H})$$

then $S_{z,0}=S_z$.

Proposition I-2.1.1. Let $\mathscr{H}$ contain the constant 1 and $a \in D$.

(1) Kerτ_a is a closed subspace of $\mathscr{H}$.

(2) If $\mathscr{H}$ is an invariant subspace of T_z, then the range of T_{z-a} is a subspace of Kerτ_a and is not dense in $\mathscr{H}$.

(3) If $\mathscr{H}$ is an invariant subspace of T_z, then the range of T_{z-a} coincides with Kerτ_a if and only if $\mathscr{H}$ is an invariant subspace of $S_{z,a}$.

Proof (1) Since $|\tau_a(f)| \leq \|\tau_a\| \, \|f\|$ $(f \in \mathscr{H})$, τ_a is norm continuous.

Therefore, Kerτ_a is a closed subspace.

(2) Since Ker$\tau_a \supseteq (z-a)\mathscr{H}$ and Kerτ_a is a closed set by (1), the range of T_{z-a} is not dense in $\mathscr{H}$.

(3) If Ker$\tau_a = (z-a)\mathscr{H}$, then $\mathscr{H} = \mathbb{C} + (z-a)\mathscr{H}$, and therefore $\mathscr{H}$ is invariant in $S_{z,a}$. Conversely, if $\mathscr{H}$ is $S_{z,a}$ invariant, then $\mathscr{H} = \mathbb{C} + (z-a)\mathscr{H}$, and therefore Ker$\tau_a = (z-a)\mathscr{H}$. (End of proof)

Proposition I-2.1.2. Suppose $\mathscr{H}$ contains the constant 1.

(1) If $\mathscr{H}$ is an invariant subspace of T_z, then T_z is not surjective on $\mathscr{H}$.

(2) If $\mathscr{H}$ is an invariant subspace of S_z, then S_z is surjective if and only if $\mathscr{H}$ is an invariant subspace of T_z.

Proof (1) If T_z is surjective, then there exists $g \in \mathscr{H}$ such that $1 = zg$. This implies that $1/z \in H(D)$, which is a contradiction.

(2) If S_z is surjective, then for any $f \in \mathscr{H}$, there exists $g \in \mathscr{H}$ such that $f = S_z g$. Therefore, $g = g(0) + zf$ and $g(0) \in \mathscr{H}$, so $T_z f \in \mathscr{H}$. Conversely, if $T_z \mathscr{H} \subset \mathscr{H}$, then for any $f \in \mathscr{H}$, if $g = zf$, then $g \in \mathscr{H}$ and $S_z g = f$. (End of proof)

If $a, b \in D$ and $a \neq b$, there exists $f \in \mathscr{H}$ such that $f(a) \neq f(b)$, and if τ_a is bounded on $\mathscr{H}$ for all $a \in D$, then $\mathscr{H}$ is called a **reproducing kernel Hilbert space**.

Proposition I-2.1.3. Let $\mathscr{H}$ be a reproducing kernel Hilbert space. If $\mathscr{H}$ is an invariant subspace of T_z or S_z, then T_z or S_z is bounded on $\mathscr{H}$, respectively. Also, a multiplier on $\mathscr{H}$ is bounded on $\mathscr{H}$.

Proof If f, F and f_n in $\mathscr{H}$ satisfy both $f_n \to f$ and $T_z f_n \to F$ in $\mathscr{H}$, then $T_z f = F$. In fact, for any $a \in D$, τ_a is continuous, so $f_n(a) \to f(a)$ and $af_n(a) \to F(a)$. Hence, $af(a) = F(a)$ $(a \in D)$, so $T_z f = F$. The closed graph theorem (II-1.3.2) shows that T_z is bounded. The boundedness of S_z or multiplier can be shown similarly. (End of proof)

Proposition I-2.1.4. Let $\mathcal{H}$ be a separable reproducing kernel Hilbert space. Let $g_n, g \in \mathcal{H}$. Then g_n weakly converges to g in $\mathcal{H}$ if and only if $\{\|g_n\|\}$ is bounded and $g_n(z) \to g(z)$ $(z \in D)$.

Proof If g_n weakly converges to g, then $\mathcal{H}$ is a reproducing kernel Hilbert space, and therefore, for each $z \in D$, $g_n(z) \to g(z)$ $(z \in D)$ and for each $\phi \in \mathcal{H}^*$, $\sup_n |\phi(g_n)| < \infty$. Since $\mathcal{H} = \mathcal{H}^*$, by Banach-Steinhaus Theorem (II-1.3.1), $\sup_n \|g_n\| < \infty$.

Conversely, suppose $\sup_n \|g_n\| < \infty$ and $g_n(z) \to g(z)$ $(z \in D)$. By Theorem I-1.4.2, the unit ball of the Hilbert space is weakly compact. Therefore there exists a subsequence $\{g_{nj}\}$ of $\{g_n\}$ such that g_{nj} weakly converges. In this case, since $g_{nj}(z) \to g(z)$ $(z \in D)$, g_{nj} weakly converges to g. Hence, any weakly convergent subsequence of $\{g_n\}$ converges to a single g. By Theorem I-1.4.4, the weak topology of the unit ball of $\mathcal{H}$ is metrizable, so g_n weakly converges to g. (End of proof)

The fundamental Hilbert spaces $\mathcal{H}$ of analytic functions on D studied in this book are **Hardy spaces**, **Bergman spaces**, and **Dirichlet spaces**. We shall show their definitions here. We shall show later that they are Hilbert spaces.

Definition I-2.1.5. The Hardy space $H^2 = H^2(D)$ is the set of all functions f in $H(D)$ satisfing

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta / 2\pi < \infty.$$

The square root of this upper bound is denoted by $\|f\|_{H^2}$.

Definition I-2.1.6. The Bergman space $L_a^2 = L_a^2(D)$ is the set of all functions f in $H(D)$ satisfing

$$\int_D |f(re^{i\theta})|^2 r dr d\theta / \pi < \infty$$

The square root of this integral is denoted by $\|f\|_{L_a^2}$.

Definition I-2.1.7. The Dirichlet space $\mathcal{D}^2 = \mathcal{D}^2(D)$ is the set of all functions f in $H(D)$ satisfing

$$\int_D |f'(re^{i\theta})|^2 r dr d\theta / \pi < \infty.$$

For $f \in \mathscr{D}^2$,

$$\|f\|_{\mathscr{D}^2}^2 = \|f\|_{H^2}^2 + \int_D |f'(re^{i\theta})|^2 r dr d\theta / \pi$$

If $f \in H(D)$, then it can be expanded by Taylor as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in D)$.

$$\int_0^{2\pi} e^{im\theta} d\theta / 2\pi = \delta_{0m} \quad (m = 0, \pm 1, \pm 2, \cdots)$$

Using

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta / 2\pi = \sum_{n=0}^{\infty} |a_n|^2 r^{2n},$$

$$\int_0^1 2r dr \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta / 2\pi = \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2$$

and

$$\int_0^1 2r dr \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta / 2\pi = \sum_{n=0}^{\infty} n |a_n|^2.$$

Therefore, the necessary and sufficient condition for $f \in H^2$ is $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. The necessary and sufficient condition for functions belonging to $\mathscr{D}^2$ and L_a^2 can also be obtained from the above equation using Taylor coefficients. Therefore, the following inclusion relationship holds:

$$\mathscr{P} \subset \mathscr{D}^2 \subset H^2 \subset L_a^2.$$